2013 American Control Conference (ACC)
Washington, DC, USA, June 17-19, 2013

Hidden Mode Tracking Control for a Class of Hybrid Systems

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Abstract— **In this paper, we consider the tracking control problem for a class of hidden mode hybrid systems in which the mode is not available for control. The time-varying reference trajectories are given by functions that may exhibit jumps. We tackle this problem by designing a sliding mode adaptive controller for the hybrid system to track well-posed time-varying reference trajectories that may exhibit jumps, using well-established tools for stabilization of hybrid systems. The approach is illustrated with examples.**

I. Introduction

The research field of trajectory tracking has been fairly mature for continuous or discrete systems. On the other hand, the stability analysis of hybrid systems, i.e. systems with both continuous and discrete dynamics is also relatively developed. Unfortunately, general results for tracking hybrid trajectories are only available for very specific solutions, for e.g., the work in [1], for a mechanical system subject to nonsmooth impacts and [2] for the juggling problem. Hence, there has been a growing interest in developing tracking controller for hybrid systems. More recently, general results have been developed in [3] and [4], assuming full knowledge of the system states and complete control over the system's continuous and discrete dynamics. These assumptions are essential to the control framework for avoiding "peaking phenomena" (identified in [1], [3]), which occur when the reference and plant jumps do not coincide.

However, these assumptions are difficult to guarantee, due to ever present small disturbances and imperfect knowledge of the system. One such class of hybrid systems is the hidden mode hybrid system [5], [6], [7], in which the continuous dynamics is described by a finite collection of functions, each of which corresponds to a *mode* of the hybrid system, and such that the mode is unknown or *hidden* and mode transitions are autonomous, i.e., there is no direct control over the switching mechanism that triggers the discrete events. There are a large number of applications, in which it is not realistic to assume knowledge of the mode, or it is simply impractical or costly to measure the mode. This is the case, for instance, in navigation across heterogeneous terrains, manufacturing, electronics, chemical or biological processes, etc., where the addition of mode sensors add unnecessary weight or interfere with the controlled process. Even when such sensors are present, this will be the case when the mode sensors fail.

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The control of hidden mode hybrid systems has been addressed for safety control in [7], in which the hidden mode is estimated while conservatively executing a control scheme until the mode estimate is sufficiently trustworthy. However, the wait time for the estimate to converge may cause unnecessary delays, assuming that the estimate converges in the first place. Thus, the objective of this paper is to develop a control law that asymptotically track a hybrid reference trajectory, without knowledge of the hidden modes and without requiring the convergence of the mode estimate to its true value.

This paper is organized as follows. In Section II, the modeling framework and some results of stabilization of hybrid systems in the literature are provided, while in Section III, we state the tracking problem of interest. The main contribution of this paper is the design of a hidden mode controller for hybrid feedback linearizable square systems, i.e., with as many inputs as outputs, for which we present sufficient conditions for the asymptotic tracking of a given reference in Section IV. Examples are presented in Section V to demonstrate our approach on a switched and a hybrid system: car with automatic transmission [8] and actuated dynamic walker, also known as the toddler [9]. Finally, some conclusions are provided in Section VI.

II. Preliminary Material

We first summarize the notation used throughout the paper. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; is the set of $\mathbb{R}_{\geq 0}$ non-negative real numbers; $\mathbb{N}$ is the set of natural numbers including 0. Given a set A, $\bar{A}$ denotes its closure and A^c its complement. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean vector norm and $\|x\|_\infty$ the infinity norm. Given a set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $\|x\|_A := \inf_{y \in A}\|x - y\|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ (denoted $\alpha \in \mathcal{K}$) if it is continuous, zero at zero and strictly increasing and to belong to class-$\mathcal{K}_\infty$ if it belongs to class-$\mathcal{K}$ and is unbounded. The function $\max : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ returns the maximum value of its arguments.

A. Modeling Framework

We consider a hybrid system $\mathcal{H}$ of the form

$$\begin{aligned} (\dot{x}, \dot{q}) &= (f_q(x,u), 0) && (x,u) \in C_q \\ (x,q)^+ &= (g_q(x,u), \delta_q(x,u)) && (x,u) \in D_q \\ y &= h_q(x) \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the continuous state, $q \in Q := \{1, 2, \dots, N\}$ the discrete state or *mode*, $u \in U_q \subset \mathbb{R}^m$ the input and $y \in \mathbb{R}^p$ the output. For each $q \in Q$, $U_q \subset \mathbb{R}^m$

978-1-4799-0178-4/$31.00 ©2013 AACC

is the set of admissible inputs, $C_q \subset \mathbb{R}^n \times U_q$ the flow set, $D_q \subset \mathbb{R}^n \times U_q$ the jump set, $f_q : C_q \to \mathbb{R}^n$ the continuous dynamics or *flow map*, $g_q : D_q \to \mathbb{R}^n$ the discrete transition/reset map or *impulse effects*, $\delta_q : D_q \to Q$ the mode transition map and h_q the output map. The data of the hybrid system $\mathcal{H}$ is given by (C, f, D, g, δ, h) with $C := \bigcup_{q\in Q} C_q \times \{q\}$, $f := \bigcup_{q\in Q} f_q \times \{q\}$, $D := \bigcup_{q\in Q} D_q \times \{q\}$, $g := \bigcup_{q\in Q} g_q \times \{q\}$, $\delta := \bigcup_{q\in Q} \delta_q \times \{q\}$ and $h := \bigcup_{q\in Q} h_q \times \{q\}$. It also follows from (1) that on every open interval on $C_q \setminus D_q$, the mode q remains constant, while the continuous states flow according to $\dot{x} = f_q(x, u)$. A special case is when the impulse effects are absent, i.e., the reset map is the identity, in which case the system is referred to as a switched system [10].

Solutions ϕ to the hybrid system $\mathcal{H}$ are defined by hybrid arcs on hybrid time domains, which are functions defined on subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ given by the union of intervals of the form $[t_j, t_{j+1}] \times \{j\}$, $t_{j+1} \geq t_j$. Since the mode q remains constant for each j, one can associate each solution of the hybrid system $\mathcal{H}$ with a switching sequence, indexed by an initial state $\phi(0, 0) \in \mathbb{R}^n$:

$$S^{\phi(0,0)} = (t_0, q_0), (t_1, q_1), \ldots, (t_j, q_j), \ldots, (t_N, q_N), \ldots$$

in which the sequence may or may not be infinite. We may take $t_{N+1} = \infty$ in the finite case, with all further definitions and results holding. The corresponding increasing sequence of switching times is denoted as $T_S = t_0, t_1, \ldots, t_j, \ldots, t_N, \ldots$ and the switching modes is denoted as $Q_S = q_0, q_1, \ldots, q_j, \ldots, q_N, \ldots$.

Moreover, if we restrict the solutions to the hybrid system $\mathcal{H}$ to a class of solutions known as *dwell-time* solutions [11], [10] such that the hybrid time domains are given by the union of intervals of the form $[t_j, t_{j+1}] \times \{j\}$, $t_{j+1} \geq t_j + \tau_D$ with dwell-time $\tau_D > 0$, we denote the resulting switching sequence, strictly increasing sequence of switching times and switching modes as $S^{\phi(0,0)}_{\tau_D}$, $T_{S_{\tau_D}}$ and $Q_{S_{\tau_D}}$, respectively.

B. Stability

Given an initial state $\phi(0, 0) \in \mathbb{R}^n$, $\mathcal{S}_{\mathcal{H}}(\phi(0, 0))$ denotes the set of maximal solutions ϕ to $\mathcal{H}$ with $\phi(0, 0)$.

Definition 1 (Stability [3]): A set $A \subset \mathbb{R}^n$ is said to be

1) *uniformly globally stable* if there exists $\alpha \in \mathcal{K}_\infty$ such that each solution $\phi \in \mathcal{S}_{\mathcal{H}}(\phi(0, 0))$ satisfies $\|\phi(t, j)\|_A \leq \alpha(\|\phi(0, 0)\|_A)$ for all $(t, j) \in \operatorname{dom} \phi$;
2) *uniformly globally attractive* if for each $\epsilon > 0$ and $\lambda > 0$, there exists $T > 0$ such that, for any solution, $\phi \in \mathcal{S}_{\mathcal{H}}(\phi(0, 0)$ with $\|\phi(0, 0)\|_A \leq \lambda$, $(t, j) \in \operatorname{dom} \phi$ and $t + j \geq T$ imply $\|\phi(t, j)\|_A \leq \epsilon$;
3) *uniformly asymptotically stable* if it is both uniformly globally stable and uniformly globally attractive.

Definition 2 (Lyapunov-like function candidate): Given a strictly increasing sequence of times $T_{S_{\tau_D}}$ belonging to a switching sequence $S^{\phi(0,0)}_{\tau_D}$ and a closed set of design equilibrium points defined in the output space $A \subset \mathbb{R}^p$, a function $V_q : \operatorname{dom} V_q \to \mathbb{R}$ is a Lyapunov-like function for function f_q with respect to set A on C_q over $T_{S_{\tau_D}}$ if

1) V_q is positive definite on C_q with respect to set A,
2) $\dot{V}_q(x, u) \leq 0$ for all $(x, u) \in C_q$.

Definition 3 (Sequence nonincreasing condition): If there exist candidate Lyapunov-like functions V_q as defined in Definition 2 corresponding to f_q for all $q \in Q$, we say they satisfy the sequence nonincreasing condition for a trajectory $x(\cdot)$ over a strictly increasing sequence of times $T_{S_{\tau_D}}$ belonging to a given switching sequence $S^{\phi(0,0)}_{\tau_D}$ if

$$V_q[k + 1] < V_q[k] \qquad \forall k \in \mathbb{N} \tag{2}$$

where $V_q[k]$ is defined as the infimum of all the values taken by V_q during the k-th time interval over which $(x, u) \in C_q$.

The following result for asymptotic stability of closed sets provides the main tool in guaranteeing the asymptotic tracking property of the hidden mode tracking control.

Theorem 1 (Multiple Lyapunov functions [12]): Given a switching sequence S and a closed set of design equilibrium points defined in the output space $A \subset \mathbb{R}^p$, if the *sequence nonincreasing condition* given in Definition 3 is satisfied, then the hybrid system $\mathcal{H} = (C, f, D, g, h)$ is uniformly asymptotically stable with respect to A.

Remark 1: Theorem 1 is less restrictive than the asymptotic stability conditions given in, for instance, Goebel et al. [11] and Hespanha [13], as they strictly require a nonincreasing V_q at every jump of the switching sequence $S^{\phi(0,0)}$.

By construction, the closed set of design equilibrium points defined in the output space $A \subset \mathbb{R}^p$ must be invariant during jumps, i.e. such that for all states in the jump set corresponding to points in the equilibrium set $\{x_d : y_d = h_q(x_d) \in A, (x_d, u) \in D_q\}$ for all $q \in \limsup S^{\phi(0,0)}$, there must exist $u \in U_q$ such that $h_q(g_q(x_d, u)) \in A$ (for uncontrolled impulse maps, i.e. $x_d^+ = g_q(x_d)$, $h_q(g_q(x_d)) \in A$). Note that this condition should always be checked, especially for time-varying hybrid reference trajectories.

The following lemmas present existence conditions for fulfilling the sequence nonincreasing condition of Theorem 1.

Lemma 1: Given a hybrid system $\mathcal{H} = (C, f, D, g, h)$, a switching mode sequence Q_S and a sequence of states associated with the mode switches, there exists a dwell time $\tau_D > 0$ such that the sequence nonincreasing condition holds for a dwell-time switching sequence $S^{\phi(0,0)}_{\tau_D}$.

Lemma 2: Conversely, given a dwell-time switching sequence $S^{\phi(0,0)}_{\tau_D}$ and a sequence of states associated with the mode switches, if some $u \in U_q$ for all $q \in S^{\phi(0,0)}_{\tau_D}$ exist and can be determined such that on the jump maps g_q of the hybrid system $\mathcal{H} = (C, f, D, g, h)$, $V_{q^+}(g_q(x, u), u) - V_q(x, u) \leq \alpha(\tau_D)$ for all $(x, u) \in D_q$ where $\alpha \in \mathcal{K}$, then the sequence nonincreasing condition holds.

III. Problem Statement

In this section, we state the class of hybrid systems for which we construct a feedback controller that guarantees asymptotic tracking of a given well-posed hybrid reference trajectory in the presence of bounded exogeneous disturbances. We also state assumptions that are implicitly made when defining this class of hybrid system and the conditions for well-posedness of the hybrid reference trajectory. Specifically, we consider square, hidden mode hybrid systems that are feedback linearizable.

A square hybrid system refers to a system with as many inputs as outputs, modeled as a hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}^{n_s}$, mode $q \in Q := \{1, 2, \ldots, N\}$, input $u \in \mathbb{R}^{m_s}$, and output $y \in \mathbb{R}^{m_s}$. We shall also restrict our attention to systems that are feedback linearizable to the following form: a hybrid system with state $\boldsymbol{\xi} \in \mathbb{R}^{n_p}$, composed of $\xi \in \mathbb{R}^{m_s}$ and their first $(m-1)$ derivatives, mode $q \in Q := \{1, 2, \ldots, N\}$, input $u \in \mathbb{R}^{m_s}$, and output $y = \xi$ ($n_p = m_s \times m$) given by

$$\begin{aligned} (\xi^{(m)}, \dot{q}) &= (f_q(\xi, \ldots, \xi^{(m-1)}, u), 0) \qquad && (\boldsymbol{\xi}, u) \in C_q \\ (\boldsymbol{\xi}, q)^+ &= (g_q(\boldsymbol{\xi}, u), \delta_q(\boldsymbol{\xi}, u)) && (\boldsymbol{\xi}, u) \in D_q. \end{aligned}$$

The reader is referred to the literature (e.g. [14], [15]) on feedback linearization, and specifically input-output linearization for more details. Moreover, we assume that the input $u \in \mathbb{R}^{m_s}$ has a Lebesgue integrable part u_1 and an impulsive part u_2, f_q is continuous and g_q is Lipschitz continuous. We also consider a bounded exogenous disturbance ($\|d\|_\infty \leq d_{max}$) that affects the system flow dynamics.

Note that the above system can also be represented by (henceforth known as the plant $\mathcal{H}_p$):

$$\begin{aligned} (a_1 \xi^{(m)}, \dot{q}) = (&\sum_{i=2}^{n_c - 1} \varphi_i(\xi, \ldots, \xi^{(m-1)}) a_i + \varphi_c + d \\ &+ \nu(\xi, \ldots, \xi^{(m-1)}, u_1), 0), \quad (\boldsymbol{\xi}, u_1) \in C \qquad (3) \\ (\boldsymbol{\xi}, q)^+ = (&g_q(\boldsymbol{\xi}, u_2), \delta_q(\boldsymbol{\xi})), \qquad \boldsymbol{\xi} \in D_q \end{aligned}$$

where $\nu : \mathbb{R}^{m_s} \to \mathbb{R}^{m_s} \times \mathbb{R}^{n_p}$ is the generalized input, which by construction (see for e.g. [15] for ensuring this in the process of input-output linearization), has an inverse function $\nu^{-1} : \mathbb{R}^{m_s} \times \mathbb{R}^{n_p} \to \mathbb{R}^{m_s}$ almost everywhere, such that the control input $u_1 \in \mathbb{R}^{m_s}$ can be determined. The above construction also implies an implicit assumption that the system order for each mode is the same and that u_1 can be uniquely determined without the knowledge of the mode q, which will be justified in Section IV-B.

On the other hand, φ_c represents features that are present in all modes, while for all $i \in \{1, 2, \ldots, n_c\}$, φ_i are features that are common among each mode q, and $a_i \in \{a_{i,0} = 0, a_{i,1}, \ldots, a_{i,q}, \ldots, a_{i,N}\}$ are premultipliers of the features that, similar to indicator functions, are constant when the system is in each time interval over which the mode is $q \in Q$ (when a feature does not appear in a particular mode, $a_i = a_{i,0}$). By construction, we assume that $a_{1,q} > 0, \ \forall q \in Q$.

By hidden mode hybrid systems, we refer to hybrid systems in which the mode q is unknown, either by choice or otherwise. For the plant (3), this means that the values of a_i are unknown. In addition, hidden mode hybrid systems are systems with *autonomous switching*, i.e. with no direct control over the switching mechanism that triggers the discrete events ($\delta_q(\boldsymbol{\xi}, u) = \delta_q(\boldsymbol{\xi})$ and $\boldsymbol{\xi} \in D_q$) as in the case of state-dependent switchings such as impacts and contact in mechanical systems. The ability to control switching would imply the knowledge of the mode, which is contrary to the assumption of a hidden mode system. Note that this is a harder problem than in Sanfelice et al. [3] because without knowledge of the mode the hybrid system is in, there is no possibility of guaranteeing that the jumps of the reference and plant trajectories occur simultaneously for all switching times $t_j \in T_S$ by means of controlled switching.

We consider hybrid arcs $r : \operatorname{dom} r \to \mathbb{R}^{n_p}$ defining reference trajectories to be tracked, which we assume to be given by a hybrid supervisor, which generates *well-posed* and complete dwell-time solutions to the hybrid system. Note that the well-posedness of the reference trajectories implies that the trajectories are feasible and that for every jump in the reference state $\mathbf{r} \in \mathbb{R}^{n_p}$, composed of $r \in \mathbb{R}^{m_s}$ and its $(m-1)$ derivatives, given by $\mathbf{r}^+ - \mathbf{r}$, there exists u_2 such that $\mathbf{r}^+ = g_{q_r}(\mathbf{r}, u_2)$ when $q_r \in D_{q_r}$, i.e., in the jump set of the reference trajectory, which we assume is provided by the trajectory generator alongside the reference trajectory:

$$\{u_2 \in U_{q_r} \ : \ \mathbf{r}^+ = g_{q_r}(\mathbf{r}, u_2)\} \qquad (4)$$

And if there is no impulsive input, i.e. $g_q(\boldsymbol{\xi}, u_2) = g_q(\boldsymbol{\xi})$, then the well-posedness condition imply that every jump of the hybrid reference trajectory $\mathbf{r}^+ - \mathbf{r}$ must satisfy

$$\mathbf{r}^+ = g_{q_r}(\mathbf{r}) \quad q_r \in D_{q_r} \qquad (5)$$

Finally, a complete hybrid reference trajectory $\mathbf{r}$ is one such that $\operatorname{dom} \mathbf{r}$ is unbounded.

We consider the following class of tracking hybrid controllers $\mathcal{H}_c$ with state $\eta \in \mathbb{R}^{n_c}$ (same n_c as in $\mathcal{H}_p$):

$$\begin{aligned} \dot{\eta} &= \begin{cases} f_c(\eta, \boldsymbol{\xi}, \mathbf{r}) & (\eta, \boldsymbol{\xi}, \mathbf{r}) \in C_c \\ 0 & otherwise \end{cases} \\ \eta^+ &= g_c(\eta, \boldsymbol{\xi}, r) \qquad (\eta, \boldsymbol{\xi}, \mathbf{r}) \in D_c \qquad (6) \\ u &= \kappa_c(\eta, \boldsymbol{\xi}, \mathbf{r}) \end{aligned}$$

where the flow and jump sets, as well as the flow and reset maps, C_c, D_c, f_c and g_c respectively, are defined as in (1) and $\kappa_c : \mathbb{R}^{n_c} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_c}$ is the control map. The resulting closed-loop system resulting from the interconnection of $\mathcal{H}_p$ and $\mathcal{H}_c$ is denoted $\mathcal{H}_{cl}$ and has state $x : (\boldsymbol{\xi}, \eta) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$.

We now state the the tracking control problem for the class of hybrid systems defined above.

Problem 1 (Tracking Control Problem): Given an input-output linearized, square, hidden mode hybrid system $\mathcal{H}_p$ with autonomous switching and a well-posed complete dwell-time reference trajectory r with switching sequence $S_{\tau_D}^{\phi(0,0)}$, design the controller $\mathcal{H}_c$ so that the set of points

$$A := \{\boldsymbol{\xi} \ : \ \boldsymbol{\xi}(t, j) = \mathbf{r}(t, j)\} \qquad (7)$$

is uniformly globally asymptotically stable. □

IV. Hidden Mode Tracking Control

This section presents the design of the hidden mode tracking controller. This controller is designed using sliding modes with an adaptive component to regulate the control states when the system dynamics changes from one mode to another. We show that, with suitable parameterization of the controller, this design complies with the sequence nonincreasing condition, thus guaranteeing asymptotic tracking of a given hybrid reference. Then, we consider some variations of this controller and discuss its implications as well as some issues related to implementation.

A. Controller Design

The key tool of the approach is the design of an adaptive sliding mode controller [15] for the hybrid system (3), assuming that a_i takes on values between $a_{i,min} := \min_{q\in\{0\}\cup Q} a_{i,q}$ and $a_{i,max} := \max_{q\in\{0\}\cup Q} a_{i,q}$ for all $i \in \{1, 2, \ldots, n_c\}$. Let us first define a combined error or sliding mode vector $s \in \mathbb{R}^{m_s}$

$$s = e^{(m-1)} + \lambda_{m-2}e^{(m-2)} + \ldots + \lambda_0 e = \Delta(p)e \quad (8)$$

where $\Delta(p) = p^{m-1}+\lambda_{m-2}p^{m-2}+\lambda_0$ is a stable polynomial in the Laplace variable p with parameter vector $\boldsymbol{\lambda} := \{\lambda_0, \ldots, \lambda_{m-2}\}$ and $e := \xi - r \in \mathbb{R}^{m_s}$ is the output tracking error. Note that s can be rewritten as $s = \xi^{(m-1)} - \xi_r^{(m-1)}$ where $\xi_r^{(m-1)} := r^{(m-1)} - \lambda_{m-2}e^{(m-2)} - \ldots - \lambda_0 e$.

We then proceed to solve Problem 1 by first designing $\mathcal{H}_c$ (6) such that the *sliding surface* $A_s := \{\boldsymbol{\xi} : s = 0\}$ is uniformly globally asymptotically stable. For asymptotic stability of sliding surface A_s, Theorem 1 must hold. Hence, we construct Lyapunov-like functions which are at their minima on the sliding surface, for which we prove that the sequence nonincreasing condition given in Definition 3 holds. In addition, we show that the invariance of the sliding surface is preserved during jumps. As such, when the trajectory tends towards and remains on the sliding surface, the tracking error tends to zero. In the following, we demonstrate the above claims with a few lemmas and a theorem for a given reference trajectory with dwell time:

$$\tau_D \geq \tau_{D,snc} + \tau_{D,s} \quad (9)$$

where $\tau_{D,snc}$ guarantees the sequence nonincreasing condition and $\tau_{D,s}$ is the settling time of the tracking error dynamics when $s = 0$.

Lemma 3 (Construction of Lyapunov-like functions): For each mode $q \in Q$, a Lyapunov-like function candidate V_q

$$V_q = \frac{1}{2}a_1 s^T s + \frac{1}{2}\tilde{\mathbf{a}}_q^T \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{a}}_q \quad (10)$$

can be defined with $\tilde{\mathbf{a}}_q = \hat{\mathbf{a}}_q - \mathbf{a}_q$, where $\mathbf{a}_q := [a_{1,q}, a_{2,q}, \ldots, a_{n_c,q}]^T$ and the premultiplier estimates $\hat{\mathbf{a}}_q = \eta$ are the controller states, as well as $\boldsymbol{\Gamma} = diag(\Gamma_1, \ldots, \Gamma_{n_c})$ where $\Gamma_i > 0$ for all $i \in \{1, \ldots, n_c\}$. The conditions for a Lyapunov-like function candidate (see Definition 2) are satisfied with the following control and adaptation laws:

$$\dot{\eta}_i = \begin{cases} -\Gamma_i y_i\, s & (\eta_i, \xi, r) \in C_{c,i} \\ 0 & (\eta_i, \xi, r) \in C_{c,i}^c \end{cases} \quad \forall\, i \in \{1, \ldots, n_c\}$$

$$u = \nu^{-1}(\mathbf{Y}\eta - \varphi_c - (\kappa + d_{max})\text{sgn}(s)) \quad (11)$$

where $y_i \in \mathbf{Y} := [\xi_r^{(m)}\ ,\ -\varphi_1(\cdot)\ ,\ \ldots\ ,\ -\varphi_{n_c}(\cdot)]$, the controller flow set complement is given by $C_{c,i}^c := \{(\eta_i, \xi, r) : (\eta_i \geq a_{i,max} \wedge -\Gamma_i y_i\, s \geq 0) \cup (\eta_i \leq a_{i,min} \wedge -\Gamma_i y_i\, s \leq 0)\}$, and $\kappa > 0$ is a positive constant.

Proof: For each mode q, we define a Lyapunov-like function, V_q as in (10). The derivative of the Lyapunov-like function is given by

$$\begin{aligned}\dot{V}_q &= s(a_1\xi^{(m)} - a_1\xi_r^{(m)}) + \dot{\hat{\mathbf{a}}}_q^T \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{a}}_q \\ &= s(\sum_{i=1}^{n_c}\varphi_i(\cdot)a_i + \varphi_c - a_1\xi_r^{(m)} + \nu(\cdot,u) + d) + \dot{\hat{\mathbf{a}}}_q^T \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{a}}_q \\ &= s(\nu(\cdot,u) + d - \mathbf{Y}\mathbf{a}_q - \xi_r^{(m)}) + \dot{\hat{\mathbf{a}}}_q^T \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{a}}_q,\end{aligned}$$

With the control and adaptation law in (11) for $(\eta_i, \xi, r) \in C_{c,i}$, we obtain $\dot{V}_q \leq -\kappa|s| \leq 0$. Otherwise, for $(\eta_i, \xi, r) \in C_{c,i}^c$, the adaptation is stopped ($\dot{\eta}_i = 0$). If the adaptation were not stopped, the adaptation would be such that $s^T y_i^T \tilde{a}_{i,q} + \dot{\eta}_i^T \Gamma_i^{-1}\tilde{a}_{i,q} = 0$. Since for $(\eta_i, \xi, r) \in C_{c,i}^c$, $\text{sgn}(\dot{\eta}_i) = \text{sgn}(\tilde{a}_{i,q})$, then $s^T y_i^T \tilde{a}_{i,q} \leq 0$. Therefore, stopping adaptation retains this extra negative term in $\dot{V}_q$, i.e. $\dot{V}_q \leq -\kappa|s| + s^T y_i^T \tilde{a}_{i,q} \leq -\kappa|s| \leq 0$. Moreover, V_q is positive definite. Thus, the conditions in Definition 2 hold. ∎

Lemma 4 (Sequence nonincreasing condition): The sequence nonincreasing condition holds for the multiple Lyapunov functions V_q for all $q \in S_{\tau_D}^{\phi(0,0)}$ for the switching sequence of a given reference trajectory with a dwell time given by (9) with

$$\tau_{D,snc} = \sup_{(t_j,j)\in \text{dom}\,\phi} \frac{\sqrt{2a_{q_j,1}}}{\kappa}((V_{q_{j-1}} - \Delta V_{\eta,j})^{\frac{1}{2}} - (\Delta V_j + V_{q_{j-1}} - \Delta V_{\eta,j})^{\frac{1}{2}}) \quad (12)$$

where $\Delta V_j := V_{q_j}(g_{q_{j-1}}(x,u),u) - V_{q_{j-1}}(x,u)$, $\Delta V_{\eta,j} := \frac{1}{2}\tilde{\mathbf{a}}_{q,max}^T \boldsymbol{\Gamma}^{-1}\tilde{\mathbf{a}}_{q,max}$ and $\tilde{\mathbf{a}}_{q,max} := [\max(a_{q_j,0,max} - a_{q_j,0}, a_{q_j,0} - a_{q_j,0,min}), \ldots, \max(a_{q_j,n_c-1,max} - a_{q_j,n_c-1}, a_{q_j,n_c-1} - a_{q_j,n_c-1,min})]$.

Proof: By Lemma 1, there exists of a dwell-time reference trajectory for which the sequence nonincreasing condition holds. In addition, since $\dot{V}_q \leq -\kappa|s| \leq -\kappa[\frac{2}{a_{q_j,1}}(V_q - \Delta V_{\eta,j})]^{\frac{1}{2}}$ for all $q \in Q$, we have quadratic decrease in V_q between switches. Therefore, if $\tau_{D,snc}$ is chosen as in (12) with suitable choices of $\boldsymbol{\Gamma}$ and κ, the decrease of ΔV_j for all $j \in \mathbb{N}$ such that $(t,j) \in \text{dom}\,\phi$ is guaranteed. ∎

Lemma 5 (Invariance of sliding surface): The sliding surface $s = 0$ where s is given by (8) is invariant if it satisfies Lemmas 3 and 4, the reference trajectory has a dwell time given by (9) and (12), and the impulsive control input is chosen as in (4) or if there is no impulsive control input, (5) holds.

Proof: For invariance, the sliding surface must be attractive and any trajectory starting on the sliding surface must remain on the surface. By Lemma 3, the choice of control and adaptation laws in (11) ensures the existence of multiple Lyapunov functions for the system. In addition, one can verify that $\ddot{V}_q$ is bounded almost everywhere, since s and $\tilde{\mathbf{a}}_q$ are bounded by the initial value of V_q, and remains bounded over each time interval by the sequence nonincreasing condition, while $\dot{s}$ is bounded by the closed loop system equation. Therefore, by Barbalat's Lemma, $s \to 0$ for each flow time interval, and by the multiple Lyapunov function theorem in Branicky [12], $s \to 0$ for the entire system. In fact, it can be shown that $s = 0$ is achieved in finite time [15]. This guarantees the attractiveness and the invariance of the sliding surface while the state ξ remains in its flow set. The second term of (9) ensures that that the error $\|\xi\|_A$ goes to zero within the same time interval on the flow set as when s reaches zero, since by suitable choice of $\boldsymbol{\lambda}$, (8) is Hurwitz with settling time $\tau_{D,s}$. When $s = 0$, with the impulsive control input u_2 given in (4) or if (5) holds when there is no impulsive control input, the jumps in the hybrid system coincides with that of the reference trajectory, resulting in $s(t_j, j) = s(t_j, j-1)$ for all $j > J$ where J denotes the index of the first jump for which $s = 0$, hence the sliding surface is invariant during jumps. ■

Theorem 2: The control and adaptation laws in (11) solves Problem 1 for a given reference trajectory and impulse dynamics that satisfy Lemma 4 and Lemma 5, respectively.

Proof: By Lemmas 3, 4 and 5, there exists Lyapunov-like functions for the system in Problem 1 that satisfy the conditions of Theorem 1. Thus, the sliding surface s is attractive and invariant. Furthermore, the invariance of the sliding surface s implies that $\|\xi\|_A \to 0$, since the error dynamics on the sliding surface $s = 0$ with s given (8) is stable (Hurwitz). ■

Thus, we have shown that the hidden mode tracking controller defined in (11) and (4) leads to asymptotic tracking of a given hybrid reference satisfying Lemma 4, even in the presence of bounded exogeneous disturbances. Note that nowhere in the control law is the explicit knowledge of the mode q required–hence the name.

To some extent, the adaptive component of the tracking controller is a mode observer, as it implicitly infers the mode based on the changing dynamics of the controller system. Nevertheless, this controller avoids the pitfall of many adaptive controllers because adaptation is only on a need-to-know basis. Oftentimes, the values of η_i do not converge to the true values a_i for some $i \in \{1, 2, \dots, n_c\}$, making the estimate of mode q inaccurate, or worse still, they converge to values for which the mode q is undefined. To understand the conditions for which the estimates do converge to their true values, observe that the closed loop dynamics with the control law given in (11) is the following

$$a_1 \dot{s} + \kappa s + d_{max}\text{sgn}(s) = Y\tilde{\mathbf{a}} + d.$$

Thus, as $s \to 0$, $Y\tilde{\mathbf{a}} + d \to 0$, implying that for the estimates to converge to their true values, the following must hold:

1) The disturbance d must tend to zero as $s \to 0$.
2) There exist $t_0 \geq 0, T > 0, \alpha_1 > 0$ such that, $\forall t \geq t_0,\ \int_t^{t+T} Y^T Y\ dt \geq \alpha_1 \mathbf{I}$. Note that this is the persistence of excitation condition found in literature on adaptive control (e.g. [15], [16]).
3) For all $t \in T_S$ corresponding to each $q \in \limsup S^{\phi(0,0)}$, $\tilde{\mathbf{a}}^+ - \tilde{\mathbf{a}}$ must be zero. For a possible approach to satisfy this, see the discussion on controller discrete transition maps in the following section.

Next, we discuss some variations of the control law that similarly lead to asymptotic tracking and some issues related to the implementation of the controller.

B. Variations and Implementation Issues

A variation of the control law in (11), with

$$u = \nu^{-1}(\mathbf{Y}\eta - \varphi_c - \kappa s - d_{max}\text{sgn}(s)) \tag{13}$$

can easily be shown to also lead to asymptotic tracking, since $\dot{V}_q \leq -\kappa s^T s\ \forall q \in Q$ which leads to an exponential decrease in V_q between switches (as opposed to quadratic decrease with the control law in (11)). The advantage of this variation of control law is the greater decrease in V_q when s is large, but the sliding mode is not reached in finite time, which may reinduce the peaking phenomenon. One can thus consider initially using the version in (13) and switching to the version in (11) when s is sufficiently small.

In addition, one can implement a hybrid controller by adding a discrete transition map for the controller states

$$\eta_i^+ = \eta_{i,r},\ (\eta, \boldsymbol{\xi}, \mathbf{r}) \in \{(\eta, \boldsymbol{\xi}, \mathbf{r}) : s = 0, \boldsymbol{r}^+ - \boldsymbol{r} \neq 0\} \tag{14}$$

where $\eta_{i,r}$ is the premultiplier values a_i corresponding to the mode of the given reference trajectory $\boldsymbol{r}^+$ for all $i \in \{1, \dots, n_c\}$. Since the control states is only updated with this jump map when $s = 0$, this has the effect of setting the premultiplier states to its correct values. This addition can increase the rate at which V_q decreases during flows since $\Delta V_{\eta,j}$ as defined in Lemma 4 equals zero. Thus, this may be useful especially when the given reference trajectory has a smaller dwell-time than is required for the sequence nonincreasing condition to hold for a small subset of the system modes. Note that the inclusion of this controller reset map also ensures that the third condition for mode estimate convergence is satisfied.

The presence of sgn(s) in the control laws (discontinuous across the sliding surface) leads to chattering which is undesirable in practice, because of high control activity and excitation of high frequency dynamics typically neglected in modeling. This can be achieved by smoothing out the discontinuity in a thin boundary layer around the sliding surface [15], defined by $\|s\| \leq \Phi_b$, where Φ_b is the boundary layer "thickness". Within the boundary layer, the term sgn(s) is replaced by s/Φ_b, whereas when $\|s\| > \Phi_b$, the term sgn(s) is retained. We represent this smoothed term with a saturation function sat : $\mathbb{R}^n \to \mathbb{R}^n$. However, note that with the implementation of the boundary layer, the invariance of the boundary layer may not be guaranteed during jumps, thus reinducing the peaking phenomenon, implying that the tracking can only be achieved to within a certain precision ϵ_p in the sense of [1] (rather than "perfect" tracking). Nevertheless, the hidden mode tracking controller with a sliding surface boundary layer functions sufficiently well for most applications that do not require perfect tracking. Moreover, if one is satisfied with "imperfect" tracking, the tracking controller presented in this paper can track reference trajectories with dwell times that are much smaller than the sufficiency bound given in (9). However, chattering reduction and perfect tracking may be simultaneously achievable with higher order sliding modes, or other control approaches. This is part of current research.

Finally, to implement this controller, we have to ensure that u_1 can be determined without the knowledge of q, as assumed in the problem statement in Section III. This is not unreasonable to require, as this can be resolved in the system design process, for e.g. by placing the actuator such that the input affects the system the same way in all modes or by having independent actuators for each mode such that when mode q is active, actuators corresponding to modes that are not q can be active without affecting the system dynamics. Moreover, the typical operating range of the generalized input $\nu(., u_1)$ may be different in each mode, such that the mode can be inferred and u_1 can be uniquely determined.

V. EXAMPLES

To demonstrate the effectiveness of the control scheme to asymptotically track a given trajectory with no measurement of the current mode, we present two examples: car with automatic transmission, and actuated dynamic walker, The simulations were implemented in MATLAB on a 2.2 GHz Intel Core i7 CPU.

1) Car with Automatic Transmission: We first implement the hidden mode tracking controller on an example of a switched system modeled as a finite automaton: a simplified model of a car with automatic transmission (from [8]). This switched system is given by

$$\dot{v} = -\frac{k}{m}v^2\text{sgn}(v) - g\sin\alpha + \frac{G_q}{m}\tau,$$
$$q^+ = \begin{cases} q+1 & \text{if } q \neq 4, v = \frac{1}{G_q}\omega_{high} \\ q-1 & \text{if } q \neq 1, v = \frac{1}{G_q}\omega_{low} \end{cases},$$

where $m = 1000\ kg$ is the mass of the car, $G_q = \{3, 2, 1, 0.8\}$ are the transmission gears ratios corresponding to modes/gears $q = \{1, 2, 3, 4\}$, $k = 100\ Ns^2m^{-2}$, is a constant, $\omega_{high} = 25\ rads^{-1}$ and $\omega_{low} = 15\ rads^{-1}$ are prescribed angular velocities of the engine, and α is the road inclination that is chosen to be periodic, $\alpha = 0.2\sin(\pi/2t)\ rad$.

To implement the controller outlined in Section IV, we put the plant dynamics in the form given in (3):

$$a_1\dot{v} = -a_2\frac{k}{m}v^2\text{sgn}(v) + \frac{\tau}{m} + d$$

where the hidden premultipliers/parameters are $a_1 = a_2 = 1/3$ when in gear $q = 1$, $a_1 = a_2 = 1/2$ when $q = 2$, $a_1 = a_2 = 1$ when $q = 3$, and $a_1 = a_2 = 1/0.8$ when $q = 4$, whereas the road inclination is treated as an exogeneous disturbance, $|d| \leq d_{max} = g$.

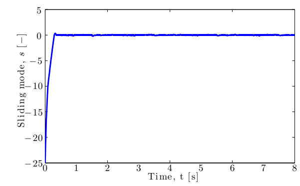
Then, the control law given by (11) can be implemented to track a reference trajectory given by another sinusoid, $v_{ref} = 10\cos(\pi t) + 15\ ms^{-1}$:

$$(\dot{\eta}_1, \dot{\eta}_2) = (-\Gamma_1\dot{v}_{ref}s, -\Gamma_2\frac{k}{m}v^2\text{sgn}(v)s),$$
$$\tau = m(\dot{v}_{ref}\eta_1 + \frac{k}{m}v^2\text{sgn}(v)\eta_2 - (\kappa + d_{max})\text{sgn}(s)),$$

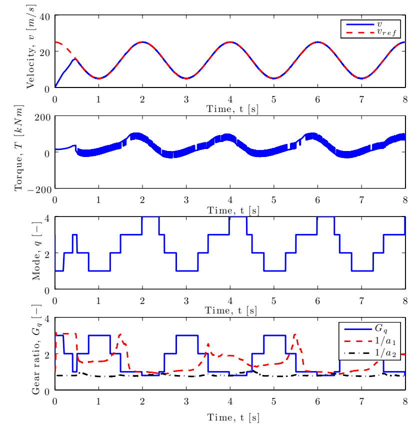
where $\Gamma_1 = \Gamma_2 = 1, \kappa = 3$, $\dot{v}_{ref} = -10\pi\sin(\pi t)$ and the sliding mode is given by $s = v - v_{ref}$.

Figure 1(a) shows the time history of the sliding mode variable, which happens to be the tracking error in this case. s converges to zero within the first half period in v, implying that we have asymptotic tracking of the reference trajectory. However, since the discontinuous version of the control law is used (without a sliding surface boundary layer), we observe chattering in the control input. Besides, as mentioned in Section IV-A, the values of η_1, η_2 do not converge to the actual values (bottom plot of Figure 1(b)). In fact, there is a conflict between the two estimates and thus, the mode estimate is undefined.

2) Actuated Dynamic Walker (Toddler): Figure 2(a) illustrates the actuated dynamic walker, also known as the toddler model [9]. The goal is to actuate the dynamic walker, such that it toddles in a periodic fashion in the frontal plane,



(a) Sliding mode variable.



(b) Plant and controller states; control input.

Fig. 1. Time history of closed-loop system states: v, η_1, η_2 and the control input τ and sliding mode variable for car with automatic transmission.

for which the dynamics is given as:

$$H(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = \tau,$$

where τ is the control input/torque generated at the hip or from ankle actuation.

When the ground contact point is in the curved portion of either foot, i.e. $|\theta| > \phi$, the dynamics are:

$$H(\theta) = I + ma^2 + mR_f^2 - 2mR_fa\cos\theta,$$
$$C(\theta, \dot{\theta}) = mR_fa\dot{\theta}\sin\theta,\ G(\theta) = mga\sin\theta,$$

whereas, when the ground contact is along the inside edge of the foot, i.e. $|\theta| \leq \phi$,

$$H(\theta) = I + ma^2 + mR_f^2 - 2mR_fa\cos\phi,$$
$$C(\theta, \dot{\theta}) = 0,\ G(\theta) = mg(a\sin\theta - R_f\sin\alpha),$$

where $\alpha = \theta - \phi$ if $\theta > 0$, otherwise $\alpha = \theta + \phi$. The mass is given by m, the moment of inertia by I and the lengths and angles are as depicted in Figure 2(a).

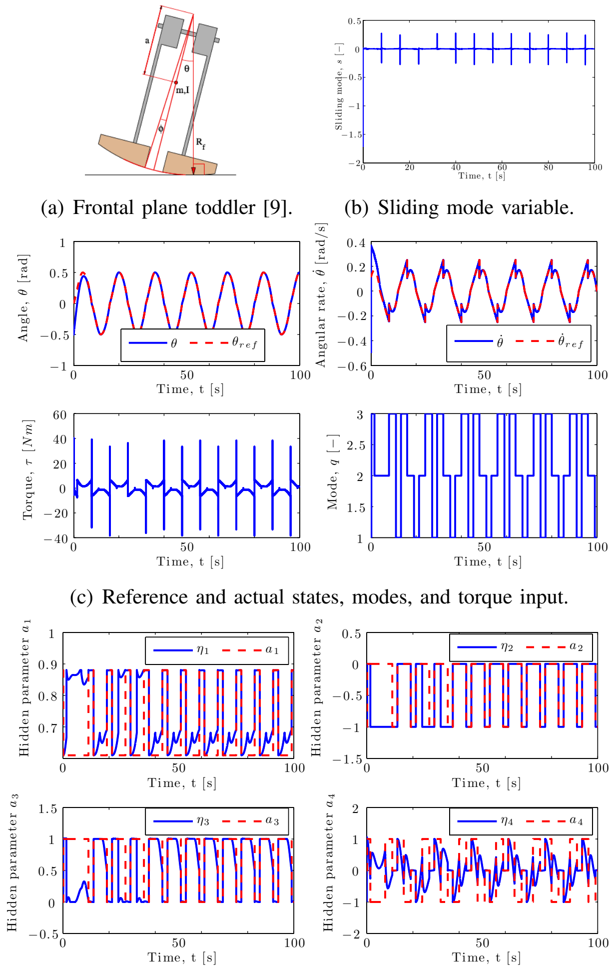
Furthermore, when $\theta = 0$, the swing leg collides with the ground, and assuming an inelastic collision, the angular rate after collision is given by

$$\dot{\theta}^+ = \dot{\theta}^-\cos\left[2\arctan\left(\frac{R_f\sin\phi}{R_f\cos\phi - a}\right)\right].$$

Thus, putting the dynamics in the form given in (3), the hybrid system describing the frontal plane toddler model is

$$\begin{aligned} a_1\ddot{\theta} &= a_2mR_fa\sin\theta\dot{\theta}^2 + a_3mgR_f\sin\theta\cos\phi + \\ &\quad a_4mgR_f\cos\theta\sin\phi - mga\sin\theta + \tau + d, \qquad (15) \\ &\quad (\theta, \dot{\theta}, \tau) \in C := \{(\theta, \dot{\theta}, \tau)\ :\ \theta \neq 0\} \\ \dot{\theta} &= \dot{\theta}\cos\left[2\arctan\left(\frac{R_f\sin\phi}{R_f\cos\phi - a}\right)\right], \\ &\quad (\theta, \dot{\theta}, \tau) \in D_q := \{(\theta, \dot{\theta}, \tau)\ :\ \theta = 0\} \end{aligned}$$

where when $q = 1$ ($|\theta| > \phi$), $a_1 = I + ma^2 + mR_f^2$, $a_2 = -1$,



(a) Frontal plane toddler [9].

(b) Sliding mode variable.

(c) Reference and actual states, modes, and torque input.

(d) Comparison of estimated premultipliers with actual values.

Fig. 2. Frontal plane toddler model [9] and time history of closed-loop system states: $\theta, \dot{\theta}, \eta_1, \eta_2, \eta_3, \eta_4$, control input and sliding mode variable.

$a_3 = 0$, and $a_4 = 0$, whereas when $q = 2$ $(0 < \theta \leq \phi)$, $a_1 = I + ma^2 + mR_f^2 - 2mR_f a \cos\phi$, $a_2 = 0$, $a_3 = 1$, and $a_4 = -1$, and, finally, when $q = 3$ $(-\phi \leq \theta \leq 0)$, $a_1 = I + ma^2 + mR_f^2 - 2mR_f a \cos\phi$, $a_2 = 0$, $a_3 = 1$, and $a_4 = 1$. Note that $2mR_f a \cos\theta\ddot{\theta}$ term in $q = 1$ is treated as a disturbance, $|d| \leq 2mR_f a\ddot{\theta}_{max}$, where $\ddot{\theta}_{max}$ is the maximum expected $|\ddot{\theta}|$ for the entire trajectory.

For this example, we implemented the variant of the hidden mode tracking controller described in Section IV-B, i.e. with the control laws given by (11), (13) and (14) for the frontal plane toddler model with the following parameters: $a = 0.1\ m, R_f = 0.5\ m, m = 3\ kg, I = 0.1\ kgm^2, \phi = 0.45\ rad, g = 9.81\ ms^{-2}, \lambda = 2, \kappa = 35, \Phi_b = 0.05, d_{max} = 0.9\ Nm$ and $\Gamma = diag(30, 30, 30, 30)$.

Figure 2(c) shows the time history of the reference and actual hybrid system states, torque input and the controller states. The controller quickly tracks the given reference trajectory (denoted by dashed red lines). The transients disappears within two periods in θ, which we can see from the settling of system variables to periodic patterns.

We also observe that the chattering in the control input is almost eliminated with the implementation of the sliding surface boundary layer. However, for this smoothing, we pay a small price of having to accommodate small jumps in s during the transition between one foot to the other (see Figure 2(b)). Once again, the hidden parameters do not converge to their true values (see Figure 2(d)), and except for a few time instances, the mode is undefined. Therefore, we have shown that the hidden mode tracking controller works even without a reliable estimate of its hybrid mode.

VI. Conclusion

This paper presented a novel approach to track a given reference trajectory of a hybrid system in which the system mode is hidden, in the presence of exogeneous disturbances. We presented the proof for asymptotic tracking of a dwell-time reference trajectory for a particular class of hybrid systems, with a hidden mode tracking controller that does not require the convergence of its mode estimate. Applications of this result can range from mechanical to electrical to biological systems, provided that the assumptions in Section IV hold. By means of numerical examples, we illustrated the effectiveness of the hidden mode tracking controller.

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