Asymptotic Adaptive Tracking with Input Amplitude and Rate Constraints and Bounded Disturbances

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Abstract— **In this paper, we develop a direct model reference adaptive control framework for asymptotic adaptive tracking in the presence of actuator input amplitude and rate constraints for some classes of uncertain linear time-invariant systems and nonlinear systems. This framework also allows for rejection of bounded time-varying disturbances, without causing any chatter in the control input. Moreover, positive (ρ, μ)-modification is proposed to protect the control law from the actuator saturation limits. The design is model-based and ensures global asymptotic tracking for open-loop input-to-state stable systems, while an estimate of the domain of attraction is derived for local asymptotic tracking in the case of input-to-state unstable systems. The approach is illustrated with examples.**

I. Introduction

It is well known that in real control systems, uncertainties in the form of disturbance signals and dynamic perturbations are unavoidable. A mathematical model of any real system is at best an approximation of the system dynamics, as we often exclude high-frequency dynamics, nonlinearities in the modeling and time variations of system parameter due to wear-and-tear or changing environment. Furthermore, control input constraints are inevitable in most practical applications because of physical limitations of actuators. Even when there are no constraints, it may be desirable to intentionally impose limits, e.g., to avoid input chattering that can excite unmodeled dynamics or cause plant damage.

Literature Review. Control design in the presence of input saturation has been widely studied (see [1] for a chronological bibliography). However, the bulk of the research effort is on known systems with actuator saturation limits, with the exception of [2]–[12]. The idea of tracking an adaptive reference model, i.e., with modifications to the reference model dynamics to deal with control deficiencies due to control amplitude saturation, is proposed by Monopoli [2], without any formal stability proof. On the other hand, [7] provides a rigorous proof and a domain of attraction for stable/bounded tracking of linear time-invariant systems without modifications to the reference model. The combination of these two was formalized in [9], [11] for linear time-invariant systems to achieve asymptotic tracking of the adaptive reference model and they further introduced positive μ-modification to guarantee that the control amplitude will never incur saturation. The same approach is applied to nonlinear systems in Brunovsky form in [11] to achieve stable/bounded tracking in the presence of bounded disturbances. However, the results in [7], [9], [11] do not consider input rate saturation. On the other hand, [8], [10], [12] take both amplitude and rate saturation into consideration but do not explicitly construct the domain of attraction, or allow for modifications for avoiding input amplitude or rate saturation and for rejecting bounded disturbances.

Contributions. This paper extends the results of [9], [11] to obtain *asymptotic* tracking for linear time-invariant systems and nonlinear systems in Brunovsky form in the presence of *input amplitude and rate* constraints. Input rate saturation is considered by augmenting the reference signal by a higher order signal, akin to the approach in [8], [12]. Moreover, we propose an approach for rejecting bounded time-varying disturbances, by further modifying the reference model to include an error term associated with the tracking error, to get a closed-loop reference model. Positive (ρ, μ)-modification is also introduced to avoid the amplitude and rate saturation of the control. Similar to [7], [9], [11], we provide global and local stability guarantees for open-loop input-to-state stable and unstable systems, respectively. In the latter case, we provide an estimate of the domain of attraction.

II. Problem Formulation

We consider two classes of uncertain systems: linear time-invariant systems, and nonlinear systems in Brunovsky form, whose dynamics are defined in Sections III and IV. These systems are assumed to be perturbed by bounded time-varying disturbances, i.e., $|d(\cdot)| \leq d_{max}$. The control input $u \in \mathbb{R}$ to these systems is amplitude and rate limited:

$$u(t) = u_{max}\text{sat}\left(\frac{u_c(t)}{u_{max}}\right),\ \dot{u}(t) = \dot{u}_{max}\text{sat}\left(\frac{\dot{u}_c(t)}{\dot{u}_{max}}\right), \quad (1)$$

$$\text{with} \quad \overline{\sigma}\text{sat}\left(\frac{s(t)}{\overline{\sigma}}\right) = \begin{cases} s(t), & |s(t)| \leq \overline{\sigma}, \\ \overline{\sigma}\text{sgn}(s(t)), & |s(t)| > \overline{\sigma}, \end{cases} \quad (2)$$

and where $u_c(t)$ and its derivative $\dot{u}_c(t)$ represent the commanded control input, while u_{max} and $\dot{u}_{max}$ are the actuator amplitude and rate saturation levels. We denote the *control amplitude and rate deficiencies* as $\Delta u(t) := u(t) - u_c(t)$ and $\Delta \dot{u}(t) := \dot{u}(t) - \dot{u}_c(t)$; and assume that all system states x_p are accessible and consider two cases, one in which $\dot{x}_p$ is also measured and the other when $\dot{x}_p$ is not accessible.

A. Positive (ρ, μ)-modification

Motivated by [9], [11], we propose a control design modification that protects the adaptive input signal from amplitude and rate saturation. This is achieved by defining

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$u_{max}^{\delta_\mu} := u_{max} - \delta_\mu$, $\dot{u}_{max}^{\delta_\rho} := \dot{u}_{max} - \delta_\rho$, $\Delta u_c(t) := u_{max}^{\delta_\mu}\text{sat}\left(\frac{u_c(t)}{u_{max}^{\delta_\mu}}\right) - u_c(t)$ and $\Delta \dot{u}_c(t) := \dot{u}_{max}^{\delta_\rho}\text{sat}\left(\frac{\dot{u}_c(t)}{\dot{u}_{max}^{\delta_\rho}}\right) - \dot{u}_c(t)$, where $0 < \delta_\mu < u_{max}$ and $0 < \delta_\rho < \dot{u}_{max}$ are chosen constants. Then, the commanded input $u_c(t)$ and its derivative $u_c(t)$ are chosen with implicit equations:

$$u_c(t) = u_d(t) + \mu\Delta u_c(t) \quad (3)$$

$$\dot{u}_c(t) = \dot{u}_{d,\mu}(t) + \rho\Delta\dot{u}_c(t) \quad (4)$$

where $u_d(t)$ is the desired input before μ-modification and $\dot{u}_{d,\mu}(t)$ the desired input rate after μ-modification but before ρ-modification[1], which will be given for each class of problems in Sections III and IV. The following lemma gives the explicit solutions of $u_c(t)$ and $\dot{u}_c(t)$.

Lemma 1: For $\mu > 0$ and $\rho > 0$, the explicit solutions to (3) and (4) $\forall t > 0$ are given by:

$$u_c(t) = \frac{1}{1+\mu}\left(u_d(t) + \mu u_{max}^{\delta_\mu}\text{sat}\left(\frac{u_d(t)}{u_{max}^{\delta_\mu}}\right)\right) \quad (5)$$

$$\dot{u}_c(t) = \frac{1}{1+\rho}\left(\dot{u}_{d,\mu}(t) + \rho\dot{u}_{max}^{\delta_\rho}\text{sat}\left(\frac{\dot{u}_{d,\mu}(t)}{\dot{u}_{max}^{\delta_\rho}}\right)\right). \quad (6)$$

Proof: The proof for $u_c(t)$ is given in [9], [11], and the same proof applies for $\dot{u}_c(t)$. ■

Remark 1: The input amplitude and rate constraints need not be symmetric. We can similarly have asymmetric limits of $u(t)$ and $\dot{u}(t)$, as well as derive the (ρ,μ)-modified command inputs by replacing $\overline{\sigma}\text{sat}\left(\frac{s(t)}{\overline{\sigma}}\right)$ with

$$\text{asat}(s(t),\underline{\sigma},\overline{\sigma}) := \begin{cases} s(t), & \underline{\sigma} \le s(t) \le \overline{\sigma} \\ \overline{\sigma}, & s(t) > \overline{\sigma} \\ \underline{\sigma}, & s(t) < \underline{\sigma} \end{cases} \quad (7)$$

where $\underline{\sigma}$ represents either u_{min}, $\dot{u}_{min}$, $u_{min}^{\delta_\mu} := u_{min} + \delta_\mu$, or $\dot{u}_{min}^{\delta_\rho} := \dot{u}_{min} + \delta_\rho$; while $\overline{\sigma}$ represents u_{max}, $\dot{u}_{max}$, $u_{max}^{\delta_\mu} := u_{max} - \delta_\mu$ or $\dot{u}_{max}^{\delta_\rho} := \dot{u}_{max} - \delta_\rho$.

B. Closed-loop higher-order adaptive reference model

Inspired by the approach in [2], the open-loop reference model (ORM) is modified to include a control deficiency feedback, which adaptively modifies the reference model, and a tracking error feedback, which leads to a closed-loop reference model. Furthermore, a higher-order reference signal r is considered as in [8] to prevent input rate saturation. Thus, in general, the closed-loop higher-order adaptive reference model (CHARM) has the form:

$$\dot{x}_m(t) = \dot{x}_m^{ORM}(t) + a(\Delta u_d(t)) + c(e(t)) \quad (8)$$

$$\dot{r}(t) = h(r_d(t), \Delta\dot{u}_d(t)) \quad (9)$$

where $\dot{x}_m^{ORM}(t)$ is the open-loop reference model dynamics, which is modified by an adaptive term $a(\Delta u_d(t))$, as in [9], [11], as well as two novel additions, namely a tracking error feedback term $c(e(t))$, and a higher order dynamics of $r(t)$ given by $h(r_d(t), \Delta\dot{u}_d(t))$, with $r_d(t)$ being the desired reference signal of the ORM, whereas the tracking error vector and deficiencies are defined as

$$e(t) := x_p(t) - x_m(t) \quad (10)$$

$$\Delta u_d(t) := u_{max}\text{sat}\left(\frac{u_c(t)}{u_{max}}\right) - u_d(t) \quad (11)$$

$$\Delta\dot{u}_d(t) := \dot{u}_{max}\text{sat}\left(\frac{\dot{u}_c(t)}{\dot{u}_{max}}\right) - \dot{u}_{d,\mu}(t). \quad (12)$$

C. Problem Statement

Problem 1: Given an open-loop reference model (ORM), design an adaptive control signal[2], $u_c(t)$, as well as the modification terms of CHARM, i.e., the signals $c(e(t))$, $a(\Delta u_d(t))$ and $h(r_d(t), \Delta\dot{u}_d(t))$ in (8), so that the state $x_p(t)$ of an uncertain plant with input amplitude and rate constraints *asymptotically* tracks the adaptively modified reference model state $x_m(t)$, while all signals of the plant and reference model remain bounded.

Remark 2: The next two sections deal with stand-alone classes of systems and the overlap in notations is intentional in order to preserve the notations used in [9], [11].

III. Class 1: Linear Time-Invariant Systems

We first consider linear time-invariant systems [9], [11]:

$$\dot{x}_p(t) = A_p x_p(t) + b_p(\lambda u(t) + d(x_p(t), t)) \quad (13)$$

where $x_p \in \mathbb{R}$ is the system state, A_p is an unknown matrix, b_p is a known constant vector. We assume that $0 < \lambda_{min} \le \lambda \le \lambda_{max}$ and $|d(\cdot)| \le d_{max}$, with known λ_{max} and d_{max}.

Ideally, we would like to track the open loop reference model (ORM), such that $\dot{x}_m = \dot{x}_m^{ORM} := A_m x_m + b_m r_d$. However, if the ORM is infeasible due to the constraints on the control input and the presence of exogenous disturbance, modifications of the ORM in the form of the following CHARM dynamics is considered:

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m + b_m(r(t) + k_u(t)\Delta u_d(t)) \\ &\quad + \phi b_p \text{sgn}(e^T(t)Pb_p)d_{max} \\ \dot{r}_o(t) &= \dot{r}_d(t) + \Lambda_r(r(t) - r_d(t)) \\ \dot{r}(t) &= \begin{cases} \dot{r}_o(t) + \frac{1}{k_r(t)}\Delta\dot{u}_d(t), & |u_c(t)| \le u_{max}^{\delta_\mu} \\ \dot{r}_o(t) + \frac{1+\mu}{k_r(t)}\Delta\dot{u}_d(t), & u_{max}^{\delta_\mu} < |u_c(t)| \le u_{max} \\ \dot{r}_o(t), & \text{otherwise} \end{cases} \end{aligned} \quad (14)$$

where $\phi \ge 1$, $\Lambda_r < 0$, and $\Delta u_d(t)$ and $\Delta\dot{u}_d(t)$ are given by (11) and (12). The introduction of the $k_u(t)\Delta u_d(t)$ term to adaptively modify the ORM is first proposed in [2], [7], [9], [11] to deal with control input amplitude constraints, whereas the $\phi b_p \text{sgn}(e^T(t)Pb_p)d_{max}$ and $\dot{r}(t)$ terms are novel. The former term is inspired by the disturbance rejection approach in sliding mode control, but since this term is not included in the control input as is done in sliding mode control, there will be no input chattering, which is a desirable trait in many practical applications. The latter term is motivated by [8], [12] to deal with control input rate constraints, and with the ρ-parametrization presented in Section II-A, the input rate constraints can be avoided with a suitably large ρ.

[1]This definition of the desired input rate is to be distinguished from $\dot{u}_{d,o}$, which is before (ρ,μ)-modification, and $\dot{u}_d$, which is the derivative of u_d after ρ- but before μ-modifications, defined later in the paper.

[2]Note that only the commanded input $u_c(t)$ is sent to the actuator; the commanded input rate $\dot{u}_c(t)$ in (6) is an intermediate variable to obtain (12) and is not explicitly implemented.

To solve Problem 1 for this class of systems, the following assumptions are made:

Assumption 1 (Matching conditions): $\exists k_x^*, k_r^*, k_u^*$, so that

$$b_p \lambda k_x^{*T} = A_m - A_p,\ b_p \lambda k_r^* = b_m,\ b_m k_u^* = b_p \lambda \quad (15)$$

Assumption 2: There exists $R > 0$ such that $x_p \in \mathcal{B}_R := \{x_p : \|x_p\| \le R\}$ and $b_p \lambda_{min} u_{max} \ge \max_{x_p \in \mathcal{B}_R} d(x_p, t)$.

From Assumption 1 and λ_{max}, we obtain $k_r^* \ge k_{r,min} := \frac{1}{\lambda_{max}} (b_p^T b_m)(b_p^T b_p)^{-1}$. Next, control and adaptation laws are derived for two different cases, one in which the state derivative $\dot{x}_p$ is accessible and another when $\dot{x}_p$ is inaccessible:

Case 1: $\dot{\mathbf{x}}_\mathbf{p}(\mathbf{t})$ accessible. Since $\dot{x}_p(t)$ is accessible, we can choose the control and adaptation laws as:

$$u_d(t) = k_x^T(t) x_p(t) + k_r(t) r(t), \quad (16)$$

$$\dot{u}_{d,o}(t) = k_x^T(t)\dot{x}_p(t) + \dot{k}_x^T(t) x_p(t) + \dot{k}_r(t) r(t) + k_r(t)\dot{r}_o(t)$$

$$\dot{u}_{d,\mu}(t) = \begin{cases} \dot{u}_{d,o}(t), & |u_c(t)| \le u_{max}^{\delta_\mu} \\ \frac{1}{1+\mu}\dot{u}_{d,o}(t), & u_{max}^{\delta_\mu} < |u_c(t)| \le u_{max} \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

$$\dot{k}_x(t) = -\Gamma_x x_p(t) e^T(t) P b_p;\ \dot{k}_{r,o}(t) = -\gamma_r r(t) e^T(t) P b_p$$

$$\dot{k}_r(t) = \begin{cases} 0, & k_r(t) \le k_{r,min} \wedge \dot{k}_{r,o}(t) < 0 \\ \dot{k}_{r,o}(t), & \text{otherwise} \end{cases} \quad (18)$$

$$\dot{k}_u(t) = \gamma_u \Delta u_d(t) e^T(t) P b_m$$

with $u_c(t)$ and $\dot{u}_c(t)$ given in (5) and (6), and where $\Gamma_x = \Gamma_x^T \succ 0$, $\gamma_r > 0$ and $\gamma_u > 0$, while $P = P^T$ is the solution of the algebraic Lyapunov equation $A_m^T P + P A_m = -Q$ for arbitrary $Q \succ 0$. The tracking error dynamics is given by

$$\dot{e}(t) = A_m e(t) + b_p \lambda (\tilde{k}_x^T(t) x_p(t) + \tilde{k}_r(t) r(t)) \quad (19)$$
$$- b_m \tilde{k}_u(t) \Delta u_d(t) + b_p d(x_p(t), t) - \phi b_p \mathrm{sgn}(e^T(t) P b_p) d_{max}$$

where the parameter errors are defined as $\tilde{k}_x(t) := k_x(t) - k_x^*$, $\tilde{k}_r(t) := k_r(t) - k_r^*$ and $\tilde{k}_u(t) := k_u(t) - k_u^*$. Thus, a Lyapunov function for this system and its derivative are:

$$V(t) = e^T(t) P e(t) + \lambda(\tilde{k}_x(t)^T \Gamma_x^{-1} \tilde{k}_x(t) + \gamma_r^{-1} \tilde{k}_r(t)^2) + \gamma_u^{-1} \tilde{k}_u(t)^2 \quad (20)$$

$$\dot{V}(t) \le -e^T(t) Q e(t) - 2\phi e^T(t) P b_p \mathrm{sgn}(e^T(t) P b_p) d_{max} + 2 e^T(t) P b_p d(x_p, t) \le -e^T(t) Q e(t) \le 0 \quad (21)$$

since $\phi \ge 1$. This implies the boundedness of the signals $e(t)$, $\tilde{k}_x(t)$, $\tilde{k}_r(t)$ and $\tilde{k}_u(t)$, and consequently, there exists e_{max}, $\tilde{k}_x^{max}$ and $\tilde{k}_r^{max}$, such that for all $t > 0$, $\|e(t)\| < e_{max}$, $\|\tilde{k}_x(t)\| < \tilde{k}_x^{max}$, $|\tilde{k}_r(t)| < \tilde{k}_r^{max} = \alpha \tilde{k}_x^{max}$, where $\alpha := \sqrt{\gamma_r / \lambda_{min}(\Gamma_x)}$. To prove asymptotic convergence of the tracking error to zero, it is essential to additionally show the boundedness of either the plant or model states, i.e., x_p or x_m. This can be shown to be globally true for stable systems (i.e., A is Hurwitz) in the following theorem:

Theorem 1: For the plant dynamics in (13) that is stable (i.e., A is Hurwitz), and with the CHARM dynamics in (14), if Assumptions 1 and 2 hold, and $\dot{x}_p(t)$ is accessible, then

- the tracking error $e(t)$ tends to zero asymptotically.
- If $\exists t^* > 0$ such that $\forall t \ge t^*$, the amplitude and rate constraints are not violated and $d_{max} \to 0$ as $t \to \infty$, then $x_p(t)$ tends towards the open-loop reference model signal $x_m^{ORM}(t)$ as $t \to \infty$.

Proof: Since the linear time-invariant system is stable and the input $u(t)$ and disturbance $d(x_p(t), t)$ are bounded, $x_p(t)$ remains bounded. By applying Barbalat's lemma, the tracking error $e(t)$ tends asymptotically to zero. Moreover, convergence to the ORM holds because $\Lambda_r < 0$. ■

For unstable systems (i.e., A is not Hurwitz), we now show the local asymptotic convergence of the tracking error to zero, and characterize the domain of attraction of the unstable systems in Theorem 2. For this, we introduce the following notations: $\eta := |\lambda_{min}(Q) - 2\lambda \|P b_p\| \|k_x^*\|$, $\kappa := \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$, $\zeta := \frac{r_{max}}{u_{max}}$, $\check{u} := \lambda u_{max} - d_{max}$ and $\hat{u} := \lambda u_{max} + d_{max}$.

Theorem 2: For unstable plant dynamics given by (13) (with non-Hurwitz A) and CHARM dynamics given by (14), assume that Assumptions 1 and 2 hold, $\dot{x}_p(t)$ is accessible, and the minimum and maximum of the desired reference signal is such that $-r_{max} < r_d^{min} \le r_d(t) \le r_d^{max} < r_{max}$, where $r_{max} < \frac{\lambda_{min}(Q)\check{u}}{\lambda |k_r^*| \eta \kappa} - \frac{d_{max}}{\lambda}$. For a given lower and upper bound on $\dot{r}_d(t)$ such that $\dot{r}_d^{min} \le \dot{r}_d(t) \le \dot{r}_d^{max}$, the design parameter Λ_r is chosen to satisfy the upper bound given by $\Lambda_r \le -\frac{2D^{\mathrm{I},1} + \dot{r}_d^{max} - \dot{r}_d^{min}}{2 r_{max} - (r_d^{min} - r_d^{max})}$, and for arbitrary $0 < \delta_\mu < u_{max}$ and $0 < \delta_\rho < \dot{u}_{max}$, the design parameters μ and ρ are selected such that the following lower bounds are satisfied:

$$\mu > \frac{(\eta + 2\|P b_p\|(\tilde{k}_x^{max} + \|k_x^*\|))\check{u} + (\tilde{k}_r^{max} + |k_r^*|)\eta r_{max}}{\eta \delta_\mu} - 2, \quad \rho > \frac{1}{\delta_\rho}(\dot{u}_{max} + C^{\mathrm{I},1}) - 2,$$

where

$$|\dot{u}_{d,\mu}(t)| \le C^{\mathrm{I},1} := \frac{4\|P b_p\|^3 e_{max} \check{u}^2 \lambda_{max}(\Gamma_x)}{\eta} + \|P b_p\| e_{max} \gamma_r r_{max}^2 + \frac{2\check{u}\|P b_p\|(\tilde{k}_x^{max} + \|k_x^*\|)(\|A_m\| + \lambda \|b_p\| \|k_x^*\|) + \eta \|b_p\| \hat{u}}{\eta} + (\tilde{k}_r^{max} + |k_r^*|)(\dot{r}_d^{max} + |\Lambda_r|(r_{max} + r_d^{max})),$$

$$D^{\mathrm{I},1} := \frac{1+\mu}{k_{r,min}}(\dot{u}_{max} + C^{\mathrm{I},1}).$$

If the system initial condition and the initial value of the candidate Lyapunov function in (20) satisfy

- $x^T(0) P x(0) < \lambda_{min}(P) \left(\frac{2\|P b_p\|}{\eta}\check{u}\right)^2$
- $\sqrt{V(0)} < \sqrt{\frac{\lambda}{\lambda_{max}(\Gamma_x)}} \left(\frac{\lambda_{min}(Q) - |k_r^*| \eta \kappa \zeta}{2\lambda \|P b_p\| + \alpha \eta \kappa \zeta}\right)$,

then

- the adaptive system in (13), (14), (24) has bounded solutions and $|r(t)| \le r_{max}$, $\forall t > 0$,
- the tracking error $e(t)$ tends to zero asymptotically, while $x^T(t) P x(t) < \lambda_{min}(P) \left(\frac{2\|P b_p\| \check{u}}{\eta}\right)^2, \forall t > 0$,
- $|u_c(t)| \le u_{max}$ and $|\dot{u}_c(t)| \le \dot{u}_{max}$, i.e., control amplitude and rate limits are avoided $\forall t > 0$, and
- if $\exists t^* > 0$ such that $\forall t \ge t^*$, the amplitude and rate constraints are not violated and $d_{max} \to 0$ as $t \to \infty$, then $x_p(t)$ tends towards the open-loop reference model signal $x_m^{ORM}(t)$ as $t \to \infty$.

Proof: The proof of the bounds on μ, r_{max} and initial $x(0)^T P x(0)$ and $\sqrt{V(0)}$ is identical to the proof of Theorem 5.1 in [9], with $-\frac{d_{max}}{\lambda}$ added to the u_{max} term as in [11]. The lower bound on ρ can be derived in a manner similar to the bound on μ in [9], [11] and is given in Appendix A.1. On the other hand, the bound on Λ_r is imposed to ensure that $\dot{r}_(t) \leq 0$ when $r(t) \geq r_{max}$ and $\dot{r}_(t) \geq 0$ when $r(t) \leq -r_{max}$, where we have applied $|\Delta \dot{u}| \leq |\dot{u}(t)| + |\dot{u}_{d,\mu}(t)| \leq \dot{u}_{max} + C^{\mathrm{I},1}$ (See Appendix A.2). Furthermore, the convergence to the ORM holds because $\Lambda_r < 0$. ■

Case 2: $\dot{\mathbf{x}}_\mathbf{p}(\mathbf{t})$ not accessible. Given that $\dot{x}_p(t)$ is not accessible, we need to slightly modify (16) by introducing an integrator in the controller and choose the control law as:

$$\begin{aligned}\dot{u}_{d,o}(t) &= k_x^T(t)(A_m x_p(t) + b_m k_{xu}^T(t)x_p(t) + b_m k_u(t)u(t) \\ &\quad - \varphi b_p \mathrm{sgn}(\tilde{u}(t)k_x^T(t)b_p)d_{max}) + \dot{k}_x^T(t)x_p(t) \\ &\quad + \dot{k}_r(t)r(t) + k_r(t)\dot{r}_o(t) - k_{\tilde{u}}\tilde{u}(t) - e^T P b_m k_u(t)\end{aligned}$$

$$\dot{u}_{d,\mu}(t) = \begin{cases} \dot{u}_{d,o}(t), & |u_c(t)| \leq u_{max}^{\delta_\mu} \\ \frac{1}{1+\mu}\dot{u}_{d,o}(t), & u_{max}^{\delta_\mu} < |u_c(t)| \leq u_{max} \\ 0, & \text{otherwise} \end{cases} \quad (22)$$

$$\begin{aligned}\dot{u}_d(t) &= k_x^T(t)(A_m x_p(t) + b_m k_{xu}^T(t)x_p(t) + b_m k_u(t)u(t) \\ &\quad - \varphi b_p \mathrm{sgn}(\tilde{u}(t)k_x^T(t)b_p)d_{max}) + \dot{k}_x^T(t)x_p(t) \quad (23) \\ &\quad + \dot{k}_r(t)r(t) + k_r(t)\dot{r}(t) - k_{\tilde{u}}\tilde{u}(t) - e^T P b_m k_u(t)\end{aligned}$$

where $\varphi \geq 1$, $k_{xu}(t)$ is an estimate of $k_u^* k_x^*$ and $k_{\tilde{u}}$ is a constant, positive parameter. Here, we have defined $u_d^\star(t) = k_x^T(t)x_p(t) + k_r(t)r(t)$ as the "desired" input when $\dot{x}_p$ is accessible (cf. (16)), and the input error is defined as $\tilde{u}(t) := u_d(t) - u_d^\star(t)$, where u_d is obtained by integrating (23) with initial condition $u_d(0) = u_d^\star(0)$. Moreover, $u_c(t)$, $\dot{u}_c(t)$ and $\Delta u_d(t)$ are given in (5), (6), (11) and (12). On the other hand, the adaptation laws are chosen as:

$$\dot{k}_x(t) = -\Gamma_x x_p(t)e^T(t)Pb_p, \ \dot{k}_{r,o}(t) = -\gamma_r r(t)e^T(t)Pb_p$$

$$\dot{k}_r(t) = \begin{cases} 0, & k_r(t) \leq k_{r,min} \wedge \dot{k}_{r,o}(t) < 0 \\ \dot{k}_{r,o}(t), & \text{otherwise} \end{cases} \quad (24)$$

$$\begin{aligned}\dot{k}_u(t) &= \gamma_u(\Delta u_d(t)e^T(t)Pb_m - u(t)\tilde{u}(t)k_x^T(t)b_m \\ &\quad + \tilde{u}(t)e^T(t)Pb_m) \\ \dot{k}_{xu}(t) &= -\Gamma_{xu}x_p(t)\tilde{u}(t)k_x^T(t)b_m\end{aligned}$$

where $\Gamma_x = \Gamma_x^T \succ 0$, $\Gamma_{xu} = \Gamma_{xu}^T \succ 0$, $\gamma_r > 0$ and $\gamma_u > 0$, while $P = P^T$ is the solution of the algebraic Lyapunov equation $A_m^T P + PA_m = -Q$ for arbitrary $Q \succ 0$. The tracking error and input error dynamics can be written as

$$\begin{aligned}\dot{e}(t) &= A_m e(t) + b_p\lambda(\tilde{k}_x^T(t)x_p(t) + \tilde{k}_r(t)r(t)) + b_m k_u^*\tilde{u}(t) \\ &\quad - b_m\tilde{k}_u(t)\Delta u_d(t) + b_p d(x_p(t),t) - \phi b_p \mathrm{sgn}(e^T(t)Pb_p)d_{max} \\ \dot{\tilde{u}} &= k_x^T(t)(b_m\tilde{k}_{xu}^T(t)x_p(t) + b_m\tilde{k}_u(t)u(t) - b_p d(t) - k_{\tilde{u}}\tilde{u}(t) \\ &\quad - \varphi b_p \mathrm{sgn}(\tilde{u}(t)k_x^T(t)b_p)d_{max}) - e^T(t)Pb_m k_u(t) \quad (25)\end{aligned}$$

where the parameter errors are defined as $\tilde{k}_x(t) := k_x(t) - k_x^*$, $\tilde{k}_r(t) := k_r(t) - k_r^*$, $\tilde{k}_u(t) := k_u(t) - k_u^*$ and $\tilde{k}_{xu}(t) := k_{xu}(t) - k_u^* k_x^*$. A Lyapunov function and its derivative are:

$$\begin{aligned}V(t) &= e^T(t)Pe(t) + \lambda(\tilde{k}_x(t)^T\Gamma_x^{-1}\tilde{k}_x(t) + \gamma_r^{-1}\tilde{k}_r(t)^2) \\ &\quad + \gamma_u^{-1}\tilde{k}_u(t)^2 + \tilde{k}_{xu}^T(t)\Gamma_{xu}^{-1}\tilde{k}_{xu}(t) + \tilde{u}(t)^2 \quad (26)\end{aligned}$$

$$\begin{aligned}\dot{V}(t) &\leq -e^T(t)Qe(t) - 2\phi e^T(t)Pb_p \mathrm{sgn}(e^T(t)Pb_p)d_{max} \\ &\quad - 2\tilde{u}(t)k_x^T(t)b_p d(x_p(t),t) + 2e^T(t)Pb_p d(x_p(t),t) \\ &\quad - 2\varphi\tilde{u}(t)k_x^T(t)b_p \mathrm{sgn}(\tilde{u}(t)k_x^T(t)b_p)d_{max} - 2k_{\tilde{u}}\tilde{u}(t)^2 \\ &\leq -e^T(t)Qe(t) - 2k_{\tilde{u}}\tilde{u}(t)^2 \leq 0 \quad (27)\end{aligned}$$

where $\phi \geq 1$ and $\varphi \geq 1$, which implies the boundedness of the signals $e(t)$, $\tilde{k}_x(t)$, $\tilde{k}_r(t)$, $\tilde{k}_u(t)$, $\tilde{k}_{xu}(t)$ and $\tilde{u}(t)$, and consequently, there exists e_{max}, $\tilde{k}_x^{max}$, $\tilde{k}_r^{max}$, $\tilde{k}_{xu}^{max}$, $\tilde{k}_u^{max}$ and $\tilde{u}_{max}$, such that for all $t > 0$, $\|e(t)\| < e_{max}$, $|\tilde{u}(t)| < \tilde{u}_{max}$, $\|\tilde{k}_x(t)\| < \tilde{k}_x^{max}$, $\|\tilde{k}_{xu}(t)\| < \tilde{k}_{xu}^{max}$, $|\tilde{k}_r(t)| < \tilde{k}_r^{max} = \alpha\tilde{k}_r^{max}$, where $\alpha := \sqrt{\gamma_r/\lambda_{min}(\Gamma_x)}$.

Similar to the first case, global asymptotic convergence of the tracking error to zero can be shown for stable systems with Hurwitz A, with identical claims as in Theorem 1, even when $\dot{x}_p$ is inaccessible. The proof is also identical, and is omitted for brevity. For unstable systems (i.e., A is not Hurwitz), we will again show its local asymptotic tracking capability, and characterize its the domain of attraction. In the analysis, η, κ, ζ, $\check{u}$ and $\hat{u}$ are as defined in the first case.

Theorem 3: For unstable plant dynamics (13) (with non-Hurwitz A) and CHARM dynamics given by (14), assume that Assumptions 1 and 2 hold, $\dot{x}_p(t)$ is inaccessible, and the minimum and maximum of the desired reference signal is such that $-r_{max} < r_d^{min} \leq r_d(t) \leq r_d^{max} < r_{max}$, where $r_{max} < \frac{\lambda_{min}(Q)\check{u}}{\lambda|k_r^*|\eta\kappa} - \frac{d_{max}}{\lambda} - \tilde{u}_{max}$. For a given lower and upper bound on $\dot{r}_d(t)$ such that $\dot{r}_d^{min} \leq \dot{r}_d(t) \leq \dot{r}_d^{max}$, the design parameter Λ_r is chosen to satisfy the upper bound given by $\Lambda_r \leq -\frac{2D^{\mathrm{I},2} + \dot{r}_d^{max} - \dot{r}_d^{min}}{2r_{max} - (r_d^{min} - r_d^{max})}$, and for arbitrary $0 < \delta_\mu < u_{max}$ and $0 < \delta_\rho < \dot{u}_{max}$, the design parameters μ and ρ are selected to satisfy the following lower bounds:

$$\begin{aligned}\mu > &\frac{(\eta + 2\|Pb_p\|(\tilde{k}_x^{max} + \|k_x^*\|))\check{u}}{\eta\delta_\mu} + \frac{(\tilde{k}_r^{max} + |k_r^*|)r_{max}}{\delta_\mu} \\ &+ \frac{\tilde{u}_{max}}{\delta_\mu} - 2, \quad \rho > \frac{1}{\delta_\rho}(\dot{u}_{max}^{\delta_\rho} + C^{\mathrm{I},2}) - 1,\end{aligned}$$

where

$$\begin{aligned}|\dot{u}_{d,\mu}(t)| \leq C^{\mathrm{I},2} &:= \frac{4\|Pb_p\|^3 e_{max}\check{u}^2\lambda_{max}(\Gamma_x)}{\eta} + \|Pb_p\|e_{max}\gamma_r r_{max}^2 \\ &+ (\tilde{k}_x^{max} + \|k_x^*\|)[(\|A_m\| + \|b_m\|(k_{xu}^{max} + \|k_u^* k_x^*\|))\frac{2\check{u}\|Pb\|}{\eta} \\ &+ \varphi\|b_p\|d_{max}] + k_{\tilde{u}}\tilde{u}_{max} + e_{max}\|Pb_m\|(\tilde{k}_u^{max} + |k_u^*|) \\ &+ (\tilde{k}_r^{max} + |k_r^*|)(\dot{r}_d^{max} + |\Lambda_r|(r_{max} + r_d^{max})),\end{aligned}$$

$$D^{\mathrm{I},2} := \frac{1+\mu}{k_{r,min}}(\dot{u}_{max} + C^{\mathrm{I},2}).$$

If the system initial condition and the initial value of the Lyapunov function in (26) are as in Theorem 2, then the same claims apply.

Proof: The proof of the bound on ρ and Λ_r is given in Appendix B.1 and B.2, while the rest of the proof is similar to that of Theorem 2, with an additional consideration of $\tilde{u}(t)$, and are omitted due to space limitation. ■

IV. CLASS 2: BRUNOVSKY FORM NONLINEAR SYSTEMS

The same principles as for the previous class of problems can be applied for the design of an adaptive control approach for nonlinear systems in Brunovsky form:

$$x_p^{(n)}(t) = \mathbf{W}^T\mathbf{\Phi}(\mathbf{x}_p(t)) + b_p u(t) + d(\mathbf{x}_p(t), t) \quad (28)$$

where $\mathbf{x}_p(t) := [x_p, \dot{x}_p, \ldots, x_p^{(n-1)}]^T$, $\mathbf{W}$ is an unknown vector, $\mathbf{\Phi}(\mathbf{x}(t))$ is a known vector and b_p is an unknown constant where $b_{max} \geq b_p \geq b_{min} > 0$ and $|d(\cdot)| \leq d_{max}$. b_{min} and d_{max} are assumed to be known. This leads to the consideration of the following CHARM dynamics:

$$x_m^{(n)}(t) = \mathbf{k}_x^{*T}\mathbf{x}_m(t) + b_m r(t) + \hat{b}(t)\Delta u_d(t) + \phi\text{sgn}(\mathbf{e}^T(t)P\mathbf{b})d_{max}$$

$$\dot{r}_o(t) = \dot{r}_d(t) + \Lambda_r(r(t) - r_d(t)) \quad (29)$$

$$\dot{r}(t) = \begin{cases} \dot{r}_o(t) + \frac{\hat{b}(t)}{b_m}\Delta\dot{u}_d, & |u_c(t)| \le u_{max}^{\delta_\mu} \\ \dot{r}_o(t) + \frac{(1+\mu)\hat{b}(t)}{b_m}\Delta\dot{u}_d, & u_{max}^{\delta_\mu} < |u_c(t)| \le u_{max} \\ \dot{r}_o(t), & \text{otherwise} \end{cases}$$

where $\phi \ge 1$, $\mathbf{e}(t) := \mathbf{x}_p(t) - \mathbf{x}_m(t)$ is the tracking error, $\mathbf{x}_m := [x_m, \dot{x}_m, \ldots, x_m^{(n-1)}]^T$, $\mathbf{k}_x^*$ is chosen such that $A = \begin{bmatrix} 0 & I \\ \mathbf{k}_x^{*T} & \end{bmatrix}$ is Hurwitz, $\mathbf{b} = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T$ and $\Delta u_d(t)$ and $\Delta\dot{u}_d(t)$ are given by (11) and (12). As with the previous class, we consider cases when $\dot{x}_p(t)$ is and is not accessible:

Case 1: $\dot{\mathbf{x}}_\mathbf{p}(\mathbf{t})$ accessible. Given the accessibility of $\dot{x}_p(t)$, we can choose the control law as:

$$u_d(t) = \frac{(\mathbf{k}_x^{*T}\mathbf{x}_p(t) + b_m r(t) - \hat{\mathbf{W}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)))}{\hat{b}(t)} \quad (30)$$

$$\dot{u}_{d,o}(t) = \frac{1}{\hat{b}(t)}(-\dot{\hat{b}}(t)u_d(t) + \mathbf{k}_x^{*T}\dot{\mathbf{x}}_p(t) + b_m\dot{r}_o(t) - \dot{\hat{\mathbf{W}}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)) - \hat{\mathbf{W}}(t)^T\dot{\mathbf{\Phi}}(\mathbf{x}_p(t)))$$

$$\dot{u}_{d,\mu}(t) = \begin{cases} \dot{u}_{d,o}(t), & |u_c(t)| \le u_{max}^{\delta_\mu} \\ \frac{1}{1+\mu}\dot{u}_{d,o}(t), & u_{max}^{\delta_\mu} < |u_c(t)| \le u_{max} \\ 0, & \text{otherwise} \end{cases} \quad (31)$$

with $u_c(t)$ and $\dot{u}_c(t)$ given in (5) and (6). The tracking error dynamics can then be written as

$$\dot{\mathbf{e}}(t) = A\mathbf{e}(t) - \mathbf{b}(\tilde{W}^T(t)\mathbf{\Phi}(\mathbf{x}_p(t)) + \tilde{b}(t)u(t) - d(\mathbf{x}_p(t), t) + \phi\text{sgn}(\mathbf{e}^T(t)P\mathbf{b})d_{max}) \quad (32)$$

where the parameter errors are $\tilde{b}(t) = \hat{b}(t) - b_p$ and $\tilde{\mathbf{W}}(t) = \hat{\mathbf{W}}(t) - \mathbf{W}$. The adaptation laws are chosen as:

$$\dot{\hat{\mathbf{W}}}(t) = \Gamma_W\mathbf{\Phi}(\mathbf{x}_p(t))\mathbf{e}^T(t)P\mathbf{b},$$

$$\dot{\hat{b}}_o(t) = \gamma_b u_{max}\text{sat}\left(\frac{u_c(t)}{u_{max}}\right)\mathbf{e}^T(t)P\mathbf{b}$$

$$\dot{\hat{b}}(t) = \begin{cases} 0, & \hat{b}(t) \le b_{min} \wedge \dot{\hat{b}}_o(t) < 0 \\ \dot{\hat{b}}_o(t), & \text{otherwise} \end{cases} \quad (33)$$

where $\Gamma_W = \Gamma_W^T \succ 0$ and $\gamma_b > 0$, while $P = P^T$ is the solution to $A^TP + PA = -Q$ for arbitrary $Q \succ 0$.

A Lyapunov function candidate and its derivative are:

$$V(t) = \mathbf{e}^T(t)P\mathbf{e}(t) + \tilde{\mathbf{W}}(t)^T\Gamma_W^{-1}\tilde{\mathbf{W}}(t) + \gamma_b^{-1}\tilde{b}(t)^2$$

$$\dot{V}(t) \le -\mathbf{e}^T(t)Q\mathbf{e}(t) - 2\phi\mathbf{e}^T(t)P\mathbf{b}\text{sgn}(\mathbf{e}^T(t)P\mathbf{b})d_{max} + 2\mathbf{e}^T(t)P\mathbf{b}d(x_p(t), t) \le -\mathbf{e}^T(t)Q\mathbf{e}(t) \le 0 \quad (34)$$

with $\phi \ge 1$, which implies the boundedness of the signals $\mathbf{e}(t)$, $\tilde{b}(t)$ and $\tilde{\mathbf{W}}(t)$, and consequently, there exist e_{max}, $\tilde{b}_{max}$ and $\tilde{W}_{max}$, such that $\forall t > 0$, $\|\mathbf{e}(t)\| < e_{max}$, $\|\tilde{\mathbf{W}}(t)\| < \tilde{W}_{max}$ and $|\tilde{b}(t)| < \tilde{b}_{max}$.

Global asymptotic convergence of the tracking error to zero can be shown for input-to-state stable systems when $\dot{x}_p$ is accessible, with identical claims and proof as in Theorem 1, thus, is not explicitly restated for the sake of brevity. But, this does not hold when the system is input-to-state unstable. In this case, we provide in Theorem 4 the characterization of the domain of attraction for which local asymptotic tracking can be achieved. We begin by introducing the following notations: $\eta := \frac{2|b_p u_{max} - \bar{d}_{max}|}{|\lambda_{min}(Q) - 2\|P\mathbf{b}\|\|\mathbf{k}_x^*|}$ and $\kappa := \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$, where we further assume that $\|\mathbf{W}\| \le W_{max}$ and in the operating region, $\bar{d}(\mathbf{x}_p, t) := |\mathbf{W}^T\mathbf{\Phi}(\mathbf{x}_p(t)) + d(\mathbf{x}_p(t), t)| \le |\mathbf{W}^T\mathbf{\Phi}(\mathbf{x}_p(t))| + d_{max} \le \bar{d}_{max}$, $\|\mathbf{\Phi}(\mathbf{x}_p(t))\| \le \Phi_{max}$, $\|\dot{\mathbf{\Phi}}(\mathbf{x}_p(t))\| \le \dot{\Phi}_{max}$ and that the upper bounds and b_{max} are known. Thus, $|\mathbf{W}^T\mathbf{\Phi}(\mathbf{x}_p(t))| \le W_\Phi^{max} := \bar{d}_{max} - d_{max}$ and $\exists\alpha$ such that $\tilde{W}_{max} + W_{max} = \alpha\tilde{b}^{max}$. We also assume that the control input authority is greater than the disturbance input, formally stated as follows:

Assumption 3: There exists $R > 0$ such that $\mathbf{x}_p \in \mathcal{B}_R := \{\mathbf{x}_p : \|\mathbf{x}_p\| \le R\}$ and $b_{min}u_{max} \ge \max_{\mathbf{x}_p \in \mathcal{B}_R} \bar{d}(\mathbf{x}_p, t)$.

Theorem 4: For the plant dynamics given by (28) and CHARM dynamics given by (29), assume that Assumption 3 holds, $\dot{x}_p(t)$ is accessible, and the minimum and maximum of the desired reference signal is such that $-r_{max} < r_d^{min} \le r_d(t) \le r_d^{max} < r_{max}$, where $r_{max} < \frac{b_{min}\lambda_{min}(Q)\eta}{2\kappa b_m b_p} - \bar{d}_{max}$. For a given lower and upper bound on $\dot{r}_d(t)$ such that $\dot{r}_d^{min} \le \dot{r}_d(t) \le \dot{r}_d^{max}$, the design parameter Λ_r is chosen to satisfy the upper bound given by $\Lambda_r \le -\frac{2D^{\text{II},1} + \dot{r}_d^{max} - \dot{r}_d^{min}}{2r_{max} - (r_d^{min} - r_d^{max})}$, and for arbitrary $0 < \delta_\mu < u_{max}$ and $0 < \delta_\rho < \dot{u}_{max}$, the design parameters μ and ρ are selected such that the following lower bounds are satisfied:

$$\mu > \frac{\|\mathbf{k}_x^*\|\|P\mathbf{b}\|\eta + b_m r_{max} + W_\Phi^{max}}{b_{min}\delta_\mu} + \frac{u_{max}}{\delta_\mu} - 2$$

$$\rho > \frac{1}{\delta_\rho}(\dot{u}_{max} + C^{\text{II},1}) - 2$$

where

$$|\dot{u}_{d,\mu}(t)| \le C^{\text{II},1} := \frac{1}{b_{min}}[\|\mathbf{k}_x^*\|\{(\|A\| + \|\mathbf{k}_x^*\|)\lambda_{min}(P)\eta^2\|P\mathbf{b}\|^2 + (b_p u_{max} + \bar{d}_{max})\} + b_m(\dot{r}_d^{max} + |\Lambda_r|(r_{max} + r_d^{max})) + \|\Gamma_W\|\Phi_{max}^2 e_{max}\|P\mathbf{b}\| + (W_{max} + \tilde{W}_{max})\dot{\Phi}_{max}] + \frac{\gamma_b u_{max} e_{max}\|P\mathbf{b}\|}{b_{min}^2}[\|\mathbf{k}_x^*\|\sqrt{\lambda_{min}(P)}\eta\|P\mathbf{b}\| + b_m r_{max} + (W_{max} + \tilde{W}_{max})\Phi_{max}],$$

$$D^{\text{II},1} := \frac{(1+\mu)b_{max}}{b_m}(\dot{u}_{max} + C^{\text{II},1}).$$

If the system initial condition and the initial value of the Lyapunov function in (34) satisfy

- $\mathbf{x}_p^T(0)P\mathbf{x}_p(0) < \lambda_{min}(P)\eta^2\|P\mathbf{b}\|^2$
- $\sqrt{V(0)} < \sqrt{\frac{1}{\gamma_b}}\left(\frac{\lambda_{min}(Q) - 2\kappa\frac{b_m b_p}{b_{min}}\frac{r_{max} + \bar{d}_{max}}{\eta}}{2\frac{\|P\mathbf{b}\|}{b_{min}}(\|\mathbf{k}_x^*\| + b_p\alpha)}\right)$

then similar claims as in Theorem 2 holds, with $\mathbf{x}_p^T(t)P\mathbf{x}_p(t) < \lambda_{min}(P)\eta^2\|P\mathbf{b}\|^2, \forall t > 0$.

Proof: The proof of the bounds on μ, r_{max} and initial $x(0)^TPx(0)$ and $\sqrt{V(0)}$ is identical to the proof given in [11]. The lower bound on ρ can be derived in a manner similar to Appendix A.1 for Theorem 2. Finally, the bound on Λ_r is imposed to ensure that $\dot{r}_(t) \le 0$ when $r(t) = r_{max}$ and $\dot{r}_(t) \ge 0$ when $r(t) = -r_{max}$, where we have applied $|\Delta\dot{u}| \le |\dot{u}(t)| + |\dot{u}_{d,\mu}| \le \frac{2+\rho}{1+\rho}(\dot{u}_{max} + C^{\text{II},1})$ (similar derivation as Appendix A.2). Similar to Theorem 2, $\Lambda_r < 0$ implies that $x_p(t)$ tends to the ORM if the amplitude and rate constraints are not violated, and $d_{max} \to 0$ as $t \to \infty$. ∎

Case 2: $\dot{\mathbf{x}}_\mathbf{p}(\mathbf{t})$ not accessible. We consider control and adaptation laws that do not require the knowledge of $\dot{\mathbf{x}}_p(t)$:

$$\dot{u}_{d,o}(t) = -k_{\tilde{u}}\tilde{u}(t) - \mathbf{e}^T(t)P\mathbf{b}\hat{b}(t) + \frac{1}{\hat{b}(t)}\Bigg(-\dot{\hat{b}}(t)u_d^\star(t) + b_m\dot{r}_o(t) - \dot{\hat{\mathbf{W}}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)) + (\mathbf{k}_x^{*T} - \hat{\mathbf{W}}^T(t)\Phi'(\mathbf{x}_p(t))) \begin{bmatrix} \begin{bmatrix}0_{n-1} & I_{n-1}\end{bmatrix}\mathbf{x}_p \\ \hat{\mathbf{W}}^T\mathbf{\Phi}(\mathbf{x}_p(t)) + \hat{b}(t)u(t) - \varphi\text{sgn}(\tilde{u}(t)L(t))d_{max} \end{bmatrix}\Bigg)$$

$$\dot{u}_{d,\mu}(t) = \begin{cases} \dot{u}_{d,o}(t), & |u_c(t)| \le u_{max}^{\delta_\mu} \\ \frac{1}{1+\mu}\dot{u}_{d,o}(t), & u_{max}^{\delta_\mu} < |u_c(t)| \le u_{max} \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

$$\dot{u}_d(t) = -k_{\tilde{u}}\tilde{u}(t) - \mathbf{e}^T(t)P\mathbf{b}\hat{b}(t) + \frac{1}{\hat{b}(t)}\Bigg(-\dot{\hat{b}}(t)u_d^\star(t) + b_m\dot{r}(t) - \dot{\hat{\mathbf{W}}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)) + (\mathbf{k}_x^{*T} - \hat{\mathbf{W}}^T(t)\Phi'(\mathbf{x}_p(t))) \begin{bmatrix} \begin{bmatrix}0_{n-1} & I_{n-1}\end{bmatrix}\mathbf{x}_p \\ \hat{\mathbf{W}}^T\mathbf{\Phi}(\mathbf{x}_p(t)) + \hat{b}(t)u(t) - \varphi\text{sgn}(\tilde{u}(t)L(t))d_{max} \end{bmatrix}\Bigg) \tag{36}$$

$$\dot{\hat{\mathbf{W}}}(t) = \Gamma_W\mathbf{\Phi}(\mathbf{x}_p(t))\left(\mathbf{e}^T(t)P\mathbf{b} - \frac{L(t)\tilde{u}(t)}{\hat{b}(t)}\right)$$

$$\dot{\hat{b}}_o(t) = \gamma_b u_{max}\text{sat}\left(\frac{u_c(t)}{u_{max}}\right)\left(\mathbf{e}^T(t)P\mathbf{b} - \frac{L(t)\tilde{u}(t)}{\hat{b}(t)}\right)$$

$$\dot{\hat{b}}(t) = \begin{cases} 0, & \hat{b}(t) \le b_{min} \wedge \dot{\hat{b}}_o(t) < 0 \\ \dot{\hat{b}}_o(t), & \text{otherwise} \end{cases} \tag{37}$$

where $\varphi \ge 1$, $k_{\tilde{u}}$ is a constant, positive parameter, $L(t) = (\mathbf{k}_x^{*T} - \hat{\mathbf{W}}^T\mathbf{\Phi}'(\mathbf{x}_p(t)))\begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}^T$ is the last element of $\mathbf{k}_x^{*T} - \hat{\mathbf{W}}^T\mathbf{\Phi}'(\mathbf{x}_p(t))$, $\mathbf{\Phi}'(\mathbf{x}_p(t))$ is the Jacobian matrix of $\mathbf{\Phi}(\mathbf{x}_p(t))$, $\Gamma_W = \Gamma_W^T \succ 0$ and $\gamma_b > 0$, while $P = P^T$ is the solution of the algebraic Lyapunov equation $A^TP + PA = -Q$ for arbitrary $Q \succ 0$. In this case where $\dot{x}_p$ is inaccessible, we defined the "desired" input as $u_d^\star(t) = \frac{(\mathbf{k}_x^{*T}\mathbf{x}_p(t) + b_m r(t) - \hat{\mathbf{W}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)))}{\hat{b}(t)}$ (cf. (30)), and the input error as $\tilde{u}(t) := u_d(t) - u_d^\star(t)$, where u_d is obtained from integrating (36) with $u_d(0) = u_d^\star(0)$. $u_c(t)$, $\dot{u}_c(t)$, $\Delta u_d(t)$ and $\Delta\dot{u}_d(t)$ are as given in (5), (6), (11) and (12).

Then, the tracking error and input error dynamics are

$$\dot{\mathbf{e}}(t) = A\mathbf{e}(t) - \mathbf{b}(\tilde{\mathbf{W}}^T(t)\mathbf{\Phi}(\mathbf{x}_p(t)) + \tilde{b}(t)u(t) - \hat{b}(t)\tilde{u}(t) + \phi\text{sgn}(\mathbf{e}^T(t)P\mathbf{b})d_{max} - d(\mathbf{x}_p(t), t)) \tag{38}$$

$$\dot{\tilde{u}}(t) = \frac{1}{\hat{b}}L(t)(\tilde{\mathbf{W}}^T(t)\mathbf{\Phi}(\mathbf{x}_p(t)) + \tilde{b}(t)u(t) - d(\mathbf{x}_p(t), t) - \varphi\text{sgn}(\tilde{u}(t)L(t))d_{max}) - k_{\tilde{u}}\tilde{u}(t) - \mathbf{e}^T(t)P\mathbf{b}\hat{b}(t) \tag{39}$$

where the parameter errors are $\tilde{b}(t) = \hat{b}(t) - b_p$ and $\tilde{\mathbf{W}}(t) = \hat{\mathbf{W}}(t) - \mathbf{W}$. A Lyapunov function and its derivative are:

$$V(t) = \mathbf{e}^T(t)P\mathbf{e}(t) + \tilde{\mathbf{W}}(t)^T\Gamma_W^{-1}\tilde{\mathbf{W}}(t) + \gamma_b^{-1}\tilde{b}(t)^2 + \tilde{u}(t)^2$$
$$\dot{V}(t) \le -\mathbf{e}^T(t)Q\mathbf{e}(t) - 2k_{\tilde{u}}\tilde{u}(t)^2 \le 0 \tag{40}$$

since $\phi \ge 1$ and $\varphi \ge 1$, and this implies the boundedness of the signals $\mathbf{e}(t)$, $\tilde{b}(t)$, $\tilde{\mathbf{W}}(t)$, and $\tilde{u}(t)$, and there exist e_{max}, $\tilde{b}_{max}$, $\tilde{W}_{max}$ and $\tilde{u}_{max}$, such that $\forall t > 0$, $\|\mathbf{e}(t)\| < e_{max}$, $\|\tilde{\mathbf{W}}(t)\| < \tilde{W}_{max}$, $|\tilde{b}(t)| < \tilde{b}_{max}$ and $|\tilde{u}(t)| < \tilde{u}_{max}$.

Once again, global asymptotic convergence of the tracking error to zero can be shown for input-to-state stable systems when the state derivative $\dot{x}_p$ is inaccessible, with identical claims and proof as in Theorem 1, thus, is not restated for brevity. In the case that the system is input-to-state unstable, we provide in Theorem 5 the characterization of the domain of attraction for which local asymptotic tracking can be achieved. The notations η, κ are as in the previous case, and we further assume that $\|\mathbf{W}\| \le W_{max}$ and in the operating region, $\overline{d}(\mathbf{x}_p, t) := |\mathbf{W}^T\mathbf{\Phi}(\mathbf{x}_p(t)) + d(\mathbf{x}_p(t), t)| \le |\mathbf{W}^T\mathbf{\Phi}(\mathbf{x}_p(t))| + d_{max} \le \overline{d}_{max}$, $\|\mathbf{\Phi}(\mathbf{x}_p(t))\| \le \Phi_{max}$, $\|\mathbf{\Phi}'(\mathbf{x}_p(t))\| \le \Phi'_{max}$, that the upper bounds and b_{max} are known, and that Assumption 3 holds. Thus, $|\mathbf{W}^T\mathbf{\Phi}(\mathbf{x}_p(t))| \le W_\Phi^{max} := \overline{d}_{max} - d_{max}$ and $\exists\alpha$ such that $\tilde{W}_{max} + W_{max} = \alpha\tilde{b}^{max}$.

Theorem 5: For the plant dynamics given by (28) and CHARM dynamics given by (29), assume that Assumption 3 holds, $\dot{x}_p(t)$ is inaccessible, and the minimum and maximum of the desired reference signal is such that $-r_{max} < r_d^{min} \le r_d(t) \le r_d^{max} < r_{max}$, where $r_{max} < \frac{b_{min}\lambda_{min}(Q)\eta}{2\kappa b_m b_p} - \overline{d}_{max} - \frac{b_{min}}{b_m}\tilde{u}_{max}$. For a given lower and upper bound on $\dot{r}_d(t)$ such that $\dot{r}_d^{min} \le \dot{r}_d(t) \le \dot{r}_d^{max}$, the design parameter Λ_r is chosen to satisfy the upper bound given by $\Lambda_r \le -\frac{2D^{\text{II},2} + \dot{r}_d^{max} - \dot{r}_d^{min}}{2r_{max} - (r_d^{min} - r_d^{max})}$, and for arbitrary $0 < \delta_\mu < u_{max}$ and $0 < \delta_\rho < \dot{u}_{max}$, the design parameters μ and ρ are selected such that the following lower bounds are satisfied:

$$\mu > \frac{\|\mathbf{k}_x^*\|\|P\mathbf{b}\|\eta + b_m r_{max} + W_\Phi^{max}}{b_{min}\delta_\mu} + \frac{\tilde{u}_{max} + u_{max}}{\delta_\mu} - 2$$

$$\rho > \frac{1}{\delta_\rho}(\dot{u}_{max} + C^{\text{II},2}) - 2$$

where

$$|\dot{u}_{d,\mu}(t)| \le C^{\text{II},2} := \frac{1}{b_{min}}\big[(\|\mathbf{k}_x^*\| + (W_{max} + \tilde{W}_{max})\Phi'_{max})(\varphi d_{max} + \sqrt{\lambda_{min}(P)}\eta\|P\mathbf{b}\| + (W_{max} + \tilde{W}_{max})\Phi_{max} + (\tilde{b}_{max} + b_p)u_{max}) + b_m(\dot{r}_d^{max} + |\Lambda_r|(r_{max} + r_d^{max})) + \|\Gamma_W\|\Phi_{max}^2 e_{max}\|P\mathbf{b}\| + e_{max}\|P\mathbf{b}\|(b_p + \tilde{b}_{max}) + k_{\tilde{u}}\tilde{u}_{max}\big] + \frac{\gamma_b u_{max} e_{max}\|P\mathbf{b}\|}{b_{min}^2}\big[\|\mathbf{k}_x^*\|\sqrt{\lambda_{min}(P)}\eta\|P\mathbf{b}\| + b_m r_{max} + (W_{max} + \tilde{W}_{max})\Phi_{max}\big],$$

$$D^{\text{II},2} := \frac{(1+\mu)b_{max}}{b_m}(\dot{u}_{max} + C^{\text{II},2}).$$

If the system initial condition and the initial value of the candidate Lyapunov function in (40) are as given in Theorem 4, then the same claims also apply.

Proof: The proof of the bound on ρ and Λ_r is similar to Appendix B.1 and B.2. The rest of the proof is similar to that of Theorem 4, with an additional consideration of $\tilde{u}(t)$, and is omitted due to space limitation. ∎

V. Illustrative Examples

1) Example of Linear Time-Invariant System [9], [11]:

$$\dot{x}_p(t) = 0.5x_p + 2(u(t) + 0.125(\sin(0.5t) + \sin(2t)))$$

The CHARM dynamics was chosen as $\dot{x}_m(t) = -6x_m(t) + 6(r(t) + k_u(t)\Delta u_d(t)) + \text{sgn}(e^T(t)Pb)d_{max}$ with $r_d(t) = 0.7(\sin(2t) + \sin(0.4t))$. The actuator limits are $u_{max} =$

0.6 and $\dot{u}_{max} = 1.5$. Figure 1 shows the simulation of this adaptive system with the following parameters and initialization: $\delta_\mu = 0.2u_{max}$, $\delta_\rho = 0.2\dot{u}_{max}$, $d_{max} = 0.25$, $b = 2$, $\lambda_{max} = 5 \Rightarrow k_{r,min} = 0.6$, $\Lambda_r = -10$, $Q = 7.5$, $\gamma_x = 1$, $\gamma_r = 1$, $\gamma_u = 1$, $k_{\tilde{u}} = 1$, $x_p(0) = 0$, $x_m(0) = 0$, $r(0) = r_d(0)$, $k_x(0) = -1$, $k_r(0) = 1$, $k_u(0) = 1$, $k_{xu}(0) = 1$, $u_d(0) = k_x(0)x_p(0) + k_r(0)r(0)$.

2) Example of Nonlinear System (Brunovsky form) [11] :

$$\dot{x}_p(t) = \mathbf{W}^T\mathbf{\Phi}(x_p) + 2u(t) + 0.125(\sin(0.5t) + \sin(2t)),$$

$$\mathbf{W} = \left[0.2, 0.01, -1, -1, 0.5\right]^T,$$

$$\mathbf{\Phi}(\cdot) = \left[x_p, x_p^3, e^{-10(x_p+0.5)^2}, e^{-10(x_p-0.5)^2}, \sin(2x_p)\right]^T$$

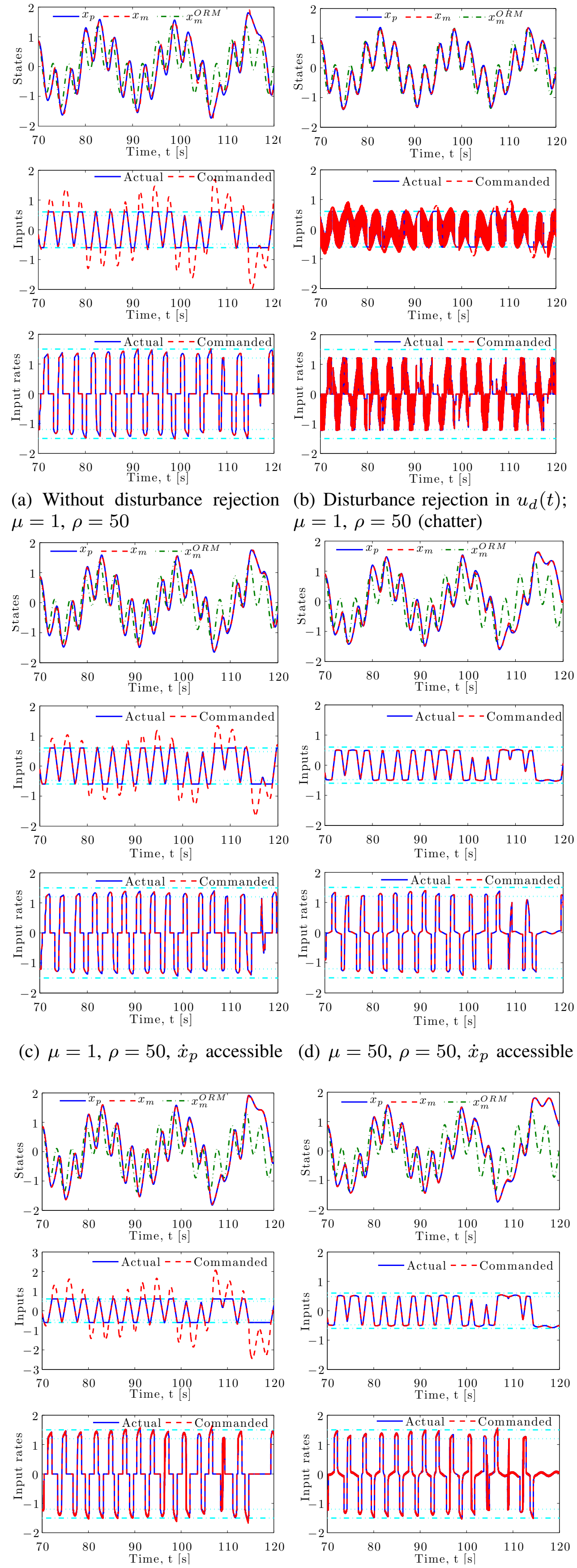
The CHARM dynamics was chosen as $\dot{x}_m(t) = -6x_m(t) + 6r(t) + \hat{b}(t)\Delta u_d(t) + \text{sgn}(e^T(t)P)d_{max}$ with the desired reference signal $r_d(t) = 0.7(\sin(2t) + \sin(0.4t))$. Figure 2 shows the simulation of this adaptive system with the following parameters and initialization: $u_{max} = 0.94$, $\dot{u}_{max} = 2$, $\delta_\mu = 0.2u_{max}$, $\delta_\rho = 0.2\dot{u}_{max}$, $d_{max} = 0.25$, $b_{min} = 1$, $\Lambda_r = -10$, $Q = 7.5$, $\Gamma_W = I_5$, $\gamma_b = 1$, $x_p(0) = 0$, $x_m(0) = 0$, $r(0) = r_d(0)$, $\hat{\mathbf{W}}(0) = \mathbf{0}$, $\hat{b}(0) = 1.5$, $u_d(0) = \frac{1}{\hat{b}(0)}(-6x_p(0) + 6r(0) - \hat{\mathbf{W}}(0)^T\mathbf{\Phi}(x_p(0)))$.

In both examples, we observe the same tracking behavior. We see from Figures 1(a) that the tracking error do not asymptotically go to zero, if disturbances are not rejected. On the other hand, if disturbance rejection is carried out by a disturbance rejection term in the control input $u(t)$ (Figures 1(b)), we have significant input chattering, which can be detrimental to many controlled systems. Note that the chattering was not limited by the rate saturation in this example, because this leads to a numerically stiff problem, which would make implementation impractical.

By the introduction of CHARM, we are able to reject bounded disturbances without any input chattering, as can be observed from Figures 1(c)-(f) and 2(a)-(b). Thus, asymptotic tracking of CHARM can be achieved in these cases without any implementation difficulties. As expected, large values of ρ and μ result in avoidance of the control saturation limits, albeit with large changes to the ORM dynamics. Moreover, when comparing the cases in which $\dot{x}_p(t)$ is accessible (Figures 1(c)-(d) and 2(a)) with the cases in which $\dot{x}_p(t)$ is not accessible (Figures 1(e)-(f) and 2(b)), we note slightly higher oscillations in the input rates, and also a marginal increase of deviation from the ORM dynamics given by $x_m^{ORM}(t)$ (green dash-dotted lines). Therefore, the knowledge of $\dot{x}_p(t)$ can be beneficial for achieving a modified trajectory that is closer to the original reference trajectory.

VI. Conclusion

This paper proposed a novel approach to asymptotically track a modified reference model (CHARM). For input-to-state stable systems with limited input amplitude and rate, we proved global tracking capability in the presence of bounded disturbances, and provided regions of attraction for input-to-state unstable systems. We also presented (ρ, μ)-modification for preventing input amplitude and rate saturation, and provided a means of disturbance rejection without



(a) Without disturbance rejection $\mu = 1$, $\rho = 50$

(b) Disturbance rejection in $u_d(t)$; $\mu = 1$, $\rho = 50$ (chatter)

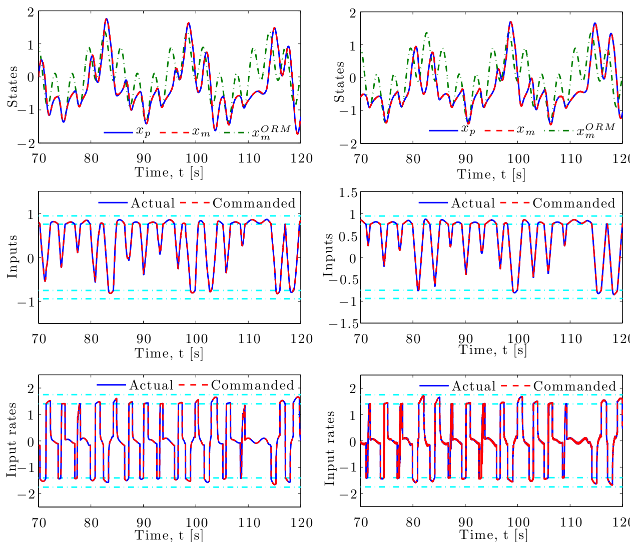
(c) $\mu = 1$, $\rho = 50$, $\dot{x}_p$ accessible

(d) $\mu = 50$, $\rho = 50$, $\dot{x}_p$ accessible

(e) $\mu = 1$, $\rho = 50$, $\dot{x}_p$ inaccessible

(f) $\mu = 50$,$\rho = 50$, $\dot{x}_p$ inaccessible

Fig. 1. States, inputs and input rates for example V-.1.



(a) $\mu = 50$, $\rho = 200$, $\dot{x}_p$ accessible (b) $\mu = 50$, $\rho = 200$, $\dot{x}_p$ inaccessible

Fig. 2. States, inputs and input rates for example V-.2.

inducing input chattering. By means of numerical examples, we illustrated the performance of this approach for uncertain linear time-invariant and nonlinear systems.

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Appendix

A. Bounds on ρ and Λ_r in Theorem 2

1) Lower Bound on ρ: As with μ, we seek a lower bound on ρ such that $\dot{u}_c(t) < \dot{u}_{max}$. From (4) and (6), we obtain

$$\Delta\dot{u}_c(t)=\frac{1}{\rho}(\dot{u}_c(t)-\dot{u}_{d,\mu}(t))=\frac{1}{1+\rho}\left(\dot{u}_{max}^{\delta_\rho}\text{sat}\left(\frac{\dot{u}_{d,\mu}(t)}{\dot{u}_{max}^{\delta_\rho}}\right)-\dot{u}_{d,\mu}(t)\right)$$

$$\Rightarrow|\Delta\dot{u}_c(t)| \leq \frac{1}{1+\rho}\left(\dot{u}_{max}^{\delta_\rho}+|\dot{u}_{d,\mu}(t)|\right) \tag{41}$$

Since $\mu > 0$, from (17),

$$\begin{aligned}|\dot{u}_{d,\mu}(t)| \leq& |\dot{k}_x^T(t)x_p(t)|+|\dot{k}_r(t)r(t)|+|k_x^T(t)\dot{x}_p(t)|+|k_r(t)\dot{r}_o(t)|\\ \leq& \|Pb_p\|e_{max}\lambda_{max}(\Gamma_x)\|x_p(t)\|^2\\ &+\|Pb_p\|e_{max}\gamma_r r_{max}^2+(\tilde{k}_x^{max}+\|k_x^*\|)\|\dot{x}_p(t)\|\\ &+(\tilde{k}_r^{max}+|k_r^*|)(\dot{r}_d^{max}+|\Lambda_r|(r_{max}+r_d^{max}))\end{aligned}$$

where $\|x_p(t)\| \leq \frac{2\|Pb_p\|\check{u}}{\eta}$ as shown in [9]. To bound $\dot{x}_p(t)$, we rewrite the system dynamics as $\dot{x}_p(t) = A_m x_p(t) - b_p\lambda k_x^{*T}x_p(t) + b_p\lambda u(t) + b_p d(x_p(t), t)$. Then,

$$\|\dot{x}_p(t)\| \leq (\|A_m\|+\lambda\|b_p\|\|k_x^*\|)2\|Pb_p\|\check{u}\frac{1}{\eta}+\|b_p\|\hat{u}$$

Thus, (41) becomes $|\Delta\dot{u}_c(t)| \leq \frac{\dot{u}_{max}^{\delta_\rho}+C^{\mathrm{I},1}}{1+\rho}$ where $C^{\mathrm{I},1}$ is defined in Theorem 2. From definition, $\Delta\dot{u}_d = \dot{u}_{max}^{\delta_\rho}\text{sat}\left(\frac{\dot{u}_c(t)}{\dot{u}_{max}^{\delta_\rho}}\right) - \dot{u}_c$. Hence, $|\dot{u}_c(t)| \leq \dot{u}_{max}^{\delta_\rho} + |\Delta\dot{u}_c(t)| \leq \dot{u}_{max}^{\delta_\rho} + \frac{\dot{u}_{max}^{\delta_\rho}+C^{\mathrm{I},1}}{1+\rho}$. In order that $|\dot{u}_c(t)| < \dot{u}_{max}, \forall t > 0$, we require $\frac{\dot{u}_{max}^{\delta_\rho}+C^{\mathrm{I},1}}{1+\rho} < \delta_\rho \Rightarrow \rho > \frac{1}{\delta_\rho}(\dot{u}_{max}^{\delta_\rho}+C^{\mathrm{I},1}) - 1 = \frac{1}{\delta_\rho}(\dot{u}_{max}+C^{\mathrm{I},1}) - 2$.

2) Upper Bound on Λ_r: From definition, $\Delta\dot{u}_d(t) = \dot{u}_{max}\text{sat}\left(\frac{\dot{u}_c(t)}{\dot{u}_{max}}\right) - \dot{u}_{d,\mu}(t)$. Thus, we can bound $|\Delta\dot{u}_d(t)| \leq \dot{u}_{max} + |\dot{u}_{d,\mu}(t)| \leq \dot{u}_{max} + C^{\mathrm{I},1}$. In order to have $|r(t)| \leq r_{max}, \forall t > 0$, we constrain $\Lambda_r < 0$ such that $\dot{r}(t) \leq 0$ when $r(t) \geq r_{max}$ and $\dot{r}(t) \geq 0$ when $r(t) \leq -r_{max}$, with the former requirement leading to $\dot{r}(t) \leq \dot{r}_d^{max} + \Lambda_r(r_{max} - r_d^{max}) + \frac{1+\mu}{k_{r,min}}(\dot{u}_{max} + C^{\mathrm{I},1}) \leq 0$ and the latter $\dot{r}(t) \geq \dot{r}_d^{min} + \Lambda_r(-r_{max} - r_d^{min}) - \frac{1+\mu}{k_{r,min}}(\dot{u}_{max} + C^{\mathrm{I},1}) \geq 0 \Rightarrow -\dot{r}_d^{min} + \Lambda_r(r_{max} + r_d^{min}) + \frac{1+\mu}{k_{r,min}}(\dot{u}_{max} + C^{\mathrm{I},1}) \leq 0$. Adding the two inequalities and rearranging, we obtain the upper bound on Λ_r given in Theorem 2.

B. Bounds on ρ and Λ_r in Theorem 3

1) Lower Bound on ρ: Since $\mu > 0$ and $|\dot{r}(t)| \leq |\dot{r}_o(t)|$, from (23),

$$\begin{aligned}|\dot{u}_{d,\mu}(t)| \leq& |\dot{k}_x^T(t)x_p(t)|+|\dot{k}_r(t)r(t)|+|k_r(t)\dot{r}(t)|+k_{\tilde{u}}|\tilde{u}(t)|\\ &+|k_x^T(t)A_m x_p(t)|+|k_x^T(t)b_m k_{xu}^T(t)x_p(t)|\\ &+\varphi|k_x^T(t)b_p|d_{max}+|e^T(t)Pb_m k_u(t)|\\ \leq& \|Pb_p\|e_{max}\lambda_{max}(\Gamma_x)\|x_p(t)\|^2+\|Pb_p\|e_{max}\gamma_r r_{max}^2\\ &+(\tilde{k}_r^{max}+|k_r^*|)(\dot{r}_d^{max}+|\Lambda_r|(r_{max}+r_d^{max}))+k_{\tilde{u}}\tilde{u}_{max}\\ &+(\tilde{k}_{max}+\|k_x^*\|)[\|A_m\|+\|b_m\|(\tilde{k}_{xu}^{max}+\|k_{xu}^*\|)\|x_p(t)\|\\ &+\varphi\|b_p\|d_{max}]+e_{max}\|Pb_m\|(\tilde{k}_u^{max}+\|k_u^*\|)\end{aligned}$$

and $\|x_p(t)\| \leq \frac{2\|Pb_p\|\check{u}}{\eta}$. Thus, (41) becomes $|\Delta\dot{u}_c(t)| \leq \frac{\dot{u}_{max}^{\delta_\rho}+C^{\mathrm{I},2}}{1+\rho}$ where $C^{\mathrm{I},2}$ is defined in Theorem 3. From definition, $\Delta\dot{u}_d = \dot{u}_{max}^{\delta_\rho}\text{sat}\left(\frac{\dot{u}_c(t)}{\dot{u}_{max}^{\delta_\rho}}\right) - \dot{u}_c$. Hence, $|\dot{u}_c(t)| \leq \dot{u}_{max}^{\delta_\rho} + |\Delta\dot{u}_c(t)| \leq \dot{u}_{max}^{\delta_\rho} + \frac{\dot{u}_{max}^{\delta_\rho}+C^{\mathrm{I},2}}{1+\rho}$. In order that $|\dot{u}_c(t)| < \dot{u}_{max}, \forall t > 0$, we require $\frac{\dot{u}_{max}^{\delta_\rho}+C^{\mathrm{I},2}}{1+\rho} < \delta_\rho \Rightarrow \rho > \frac{1}{\delta_\rho}(\dot{u}_{max}^{\delta_\rho}+C^{\mathrm{I},2}) - 1 = \frac{1}{\delta_\rho}(\dot{u}_{max}+C^{\mathrm{I},2}) - 2$.

2) Upper Bound on Λ_r: The upper bound on Λ_r can be found as in Appendix A.2 with $|\dot{u}_{d,\mu}(t)| \leq C^{\mathrm{I},2}$.