

# Asymptotically reachable states and related symmetry in systems theory

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**Abstract**—In executing state-to-state maneuvers, end states which are stabilizable states provide for robust maneuver control. In the normal circumstance that a maneuver only takes the system to a neighborhood of a stabilizable state, feedback control can be used to regulate to a neighborhood of the desired end state. In contrast, if the end state is not a stabilizable state, large try-again maneuvers which cannot be bounded by the terminal tracking error of the prior maneuver are often required in the event of a near miss. In this paper it is shown that the subspace of stabilizable states for a linear stabilizable system is the intersection of the reachable subspace and a particular controlled invariant subspace we call the constant state subspace. The stabilizable states are also the states which can be approached asymptotically with appropriate choice of control, and we use this characterization as our definition of stabilizable states.

## I. INTRODUCTION

The widespread use of computational tools for control system design has motivated the development of numerous optimal controller synthesis techniques over the past 30 years. Computationally aided synthesis has its roots in the 1960s with LQR and LQG controllers requiring numerical solutions to the differential or algebraic Riccati equation [1]. LQR and LQG along with many variations can now be placed into the more broadly applicable category of  $H_2$  design which became popular during the 1980s and 1990s along with  $H_\infty$  design and other convex optimization based approaches [2], [3]. Today, the ease with which one can obtain an optimal controller for a particular system and performance index makes it easy to overlook important performance limitations of a system.

In order to have a good understanding of the capabilities and limitations of a control system, it is essential to first examine the system for certain properties that capture important system behavior and performance limitations [2], [4], [5]. Along these lines, there is a large body of literature devoted to examining the structure of LTI systems and the related implications for control rather than focusing on the control design itself. The most widely known properties are those of controllability, reachability and their duals [6]. Another well studied property is the zero dynamics of a system [7], and yet another is the severe performance limitations imposed on nonminimum phase systems [8]. Left or right invertibility is another example of an important system property used in estimation and for establishing functional controllability [9].

The more modern approaches taken for controller synthesis usually focus on examining operator gains which is a

good way to pose the control design as an optimization, while geometric properties of subspaces tend to be more useful for identifying performance limitations and feasibility of control tasks. This paper presents what appears to be a simple yet new geometric property which, in general terms, characterizes the subspace of states which can be stabilized. This subspace was identified while trying to articulate why it is more difficult to reach certain states than others with high precision despite having a fully reachable system. The reason is that some reachable states can be regulated in the same way as the origin in a stabilizable system where a small tracking error can be driven to zero monotonically while others states, despite being reachable, require the tracking error to initially increase substantially before reaching a smaller tracking error at a later time. Additionally, in these cases where the target is not stabilizable the trajectory is forced to deviate from the desired state immediately after reaching it. The result is that the unstabilizable states require careful timing and overly robust tracking control of the reference trajectory for satisfactory performance. One example of this difficulty can be seen in the sophisticated trajectory generation techniques and careful modeling used to make state-to-state connections for acrobatic quadrotor maneuvers in [10], [11] as well as in the example in section II.

Another reason it is important to identify the subspace of asymptotically reachable states is simply for setpoint regulation. If the setpoint is chosen such that the preimage of the output map requires the state to lie outside of this subspace for perfect tracking, then controller performance will necessarily be poor. Another potential application of the proposed work is in the area of learning control. The subspace of asymptotically reachable states has an associated symmetry which may prove useful in learning control applications where the desired motion takes place near state constraints. The symmetry enables rehearsal of a desired motion elsewhere in the state space where tracking requirements are less critical.

The contributions of the paper are structured as follows. Section II provides a description of the motivating control problem. It helps to illustrate the distinction between the reachable subspace and the asymptotically reachable subspace. Section III introduces the class of systems under consideration and defines the asymptotically reachable subspace. Section IV reviews relevant background and geometric tools for analyzing LTI systems. Section V contains the primary contributions of the paper including, geometric properties of the asymptotically reachable subspace, the equivalence to the subspace of stabilizable states, a useful related symmetry, and relation to well known subspaces of LTI systems. Lastly,

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Section VI contains concluding remarks.

## II. MOTIVATING EXAMPLES

Evaluating the controllability and equivalently reachability of an LTI system is useful for determining if the origin can be stabilized, but states in the reachable subspace which can be reached in finite time are not necessarily stabilizable. Although it is a simple notion, it is important to classify the subspace of states which can be reached asymptotically as it characterizes a subspace in which point-to-point trajectories can be executed robustly. The importance of this distinction is introduced with the following examples. The first example provides intuition while the second example is what motivated the present discussion of the stabilizable subspace.

### Double integrator

A double integrator is the most simple example that can demonstrate that it may require arbitrarily large motions in the state space to make small point-to-point connections between states. Consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad (1)$$

Suppose  $(x(0), v(0))^T = (\varepsilon, 1)^T$ , with  $\varepsilon > 0$ , and the desired terminal state is  $(x(T), v(T))^T = (0, 1)^T$ . Let  $e(t) = \|(x(T), v(T))^T - (x(t), v(t))^T\|$ . Thus  $e(0) = \varepsilon$ . Since  $x(T) < x(0)$ , to drive the state to  $(0, 1)^T$  requires that at some time  $t^* \in (0, T)$ ,  $\dot{x}(t^*) < 0 \Rightarrow v(t^*) < 0$ . Thus,  $e(t^*) > 1$ . Therefore the trajectory connecting these states requires arbitrarily large motion in the sense that  $e(t^*)/e(0) > 1/\varepsilon$  is an arbitrarily large value. The large motion just described is illustrated in figure 1.

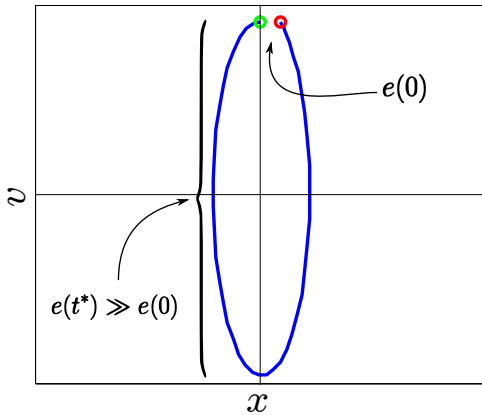


Fig. 1. State space trajectory (blue) of the system defined in (1) with a control driving the system from the initial state (red) to the desired final state (green). Notice that the point-to-point motion is small but a large state trajectory is required to connect the two states.

On the other hand suppose  $(x(0), v(0))^T$  is arbitrary and the target is  $(x(T), v(T))^T = (1, 0)^T$ . Now consider the following feedback control

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} (1/4 \ 1)}_u \begin{pmatrix} x(T) - x \\ v(T) - v \end{pmatrix}.$$

The suggested control yields a trajectory which approaches the target exponentially fast satisfying  $e(t) \leq e(0)\exp(-t/2)$ . This suggests that the latter target is better behaved in some sense. The difference between these states in this example is clear. The first target cannot be stabilized while the second can.

### VTOL sloped landing

Consider the problem of controlling a helicopter to a sloped landing site from hover. For brevity, consider a simplified model. Assume that the yaw is fixed so only pitch and roll dynamics need to be considered. Further, assume that an attitude reference tracking control stabilizes the desired attitude. Then the input to the system is the attitude and vertical velocity reference,  $(u_\phi, u_\theta, u_z)$ , and the state is the position, velocity, attitude, and attitude rate,  $(x, y, z, v_x, v_y, v_z, \phi, \theta, v_\phi, v_\theta)$ . A linearization of a low order model about hover yields the following  $(A, B)$  pair:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_x & 0 & 0 & -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_y & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_\phi^2 & 0 & -2\zeta_\phi\omega_\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega_\theta^2 & 0 & -2\zeta_\theta\omega_\theta \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \omega_\phi^2 & 0 & 0 \\ 0 & \omega_\theta^2 & 0 \end{pmatrix}. \quad (2)$$

The closed loop attitude dynamics are represented by a second order response with natural frequencies  $\omega_{\phi, \theta}$  and damping ratio  $\zeta_{\phi, \theta}$ . The lateral velocity then integrates the attitudes scaled by acceleration due to gravity,  $g$  with dissipation  $b_{x, y}$ . The lateral position is then obtained by integrating lateral velocity. The objective is to take the helicopter from an initial state  $(x_0, y_0, z_0, 0, 0, 0, 0, 0, 0, 0)$  to a sloped landing site  $(0, 0, 0, 0, 0, 0, \phi_f, \theta_f, 0, 0)$ .

It is now useful to compute the asymptotically reachable subspace for this system. This subspace will be discussed in detail in section V. Evaluating

$$\mathcal{S} = (\text{Ker}(A) + A^\dagger (\text{Im}(A) \cap \text{Im}(B))) \cap \mathcal{C},$$

where  $\mathcal{C}$  is the controllable subspace, shows that asymptoti-

cally reachable states are contained in

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Thus, any position with zero velocity, attitude, and attitude rate can be stabilized. The target landing state is clearly not within this subspace and therefore can only be reached in a finite time. Further, if the state is not reached with sufficient accuracy, an additional large motion may be required to reduce the error. With uncertainty in the model it becomes significantly more difficult to guarantee that the system will reach the target state with acceptable precision. On the other hand, if the target state were asymptotically reachable, small errors could be rejected by stabilization of the target after executing the point-to-point maneuver.

A low order model similar to the one described in (2) was used to design control laws for a Sikorsky UH60 helicopter. The model was identified from a high fidelity simulation of the helicopter made available by Aurora Flight Sciences. A LQR was used to stabilize states within the asymptotically reachable subspace. To reach the target state outside of the asymptotically reachable subspace, a trajectory and feed-forward control were generated using flatness based planning. LQR was again used, now to stabilize the trajectory. The initial state, which is contained in the asymptotically reachable subspace, was stabilized by the proposed control followed by execution of the planned trajectory.

Modeling errors between the reduced order linear model used for control design and the simulation led to trajectory tracking errors during execution of the maneuver. An unstabilizable terminal state made rejecting terminal tracking errors nontrivial and necessitated large try again maneuvers together with an iterative learning control law. Figure 2 shows the terminal output tracking error of the UH60 simulation visualized in Open Scene Graph after an initial attempt at a landing maneuver.

As this example illustrates, the goal state which is reachable from the origin cannot be stabilized and regulated by feedback control. The remainder of this paper focuses on identifying the subspace of an LTI system in which states can be stabilized.

### III. SYSTEM DESCRIPTION

The system under consideration is a continuous LTI system with input to state dynamics:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \quad (3)$$

Without loss of generality it is assumed  $B$  has full column rank.

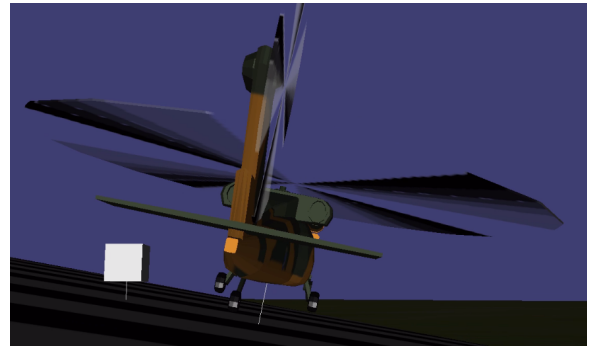


Fig. 2. An Open Scene Graph visualization of the UH60 helicopter tracking error at the planned touchdown time. The white cube represents the target position. The target attitude is illustrated by the slope of the landing site. The tracking error in position alone at the desired touchdown time is approximately two meters.

The goal of this paper is to identify the subspace of states reachable in infinite time and relate them to states stabilizable by feedback control. While it is not difficult to identify the asymptotically reachable subspace in some examples, it is useful to characterize it and its properties in a general systems setting.

**Definition 1:** *The asymptotically reachable subspace,  $\mathcal{S} \subseteq \mathbb{R}^n$ , is the subset of states such that for each  $x_0 \in \mathbb{R}^n$  and  $x_r \in \mathcal{S}$ , there exists a control signal  $u(t)$  such that a solution to (3),  $x(t)$ , passing through  $x_0$  satisfies*

$$\lim_{t \rightarrow \infty} x(t) = x_r.$$

It is subsequently shown that the asymptotically reachable subspace is equal to the subspace of states which can be stabilized by feedback control and that it is not equal to the reachable subspace in general.

### IV. RELEVANT BACKGROUND ON LTI SYSTEMS

The notion of invariant subspaces is fundamental to linear systems theory, and relevant to the present work are  $A$ -invariant subspaces,  $\mathcal{V}$ , containing the forcing subspace,  $\text{Im}(B)$ . That is,

$$A\mathcal{V} \subseteq \mathcal{V}, \quad \text{Im}(B) \subseteq \mathcal{V}. \quad (4)$$

The subspace of lowest dimension with this property is the well known reachable subspace, denoted  $\mathcal{C}$  for the remainder of the paper.

Another equally useful notion is that of a controlled invariant. A subspace  $\mathcal{X}$  is an  $(A, B)$ -controlled invariant if

$$A\mathcal{X} \subseteq \mathcal{X} + \text{Im}(B).$$

This paper will make use of a particular controlled invariant subspace,  $\mathcal{X}_c$ , referred to as the constant state subspace. This subspace consists of the states that can be held constant for a finite time interval. That is, states  $x$  for which there is an input  $u$  satisfying the equilibrium equation:

$$0 = Ax + Bu. \quad (5)$$

It follows from the definition that  $\mathcal{X}_c$  is a controlled invariant. Notice that for each  $x \in \mathcal{X}_c$ , there exists a  $u$  such that  $Ax = Bu$ . Thus  $A\mathcal{X}_c \subseteq \text{Im}(B) \subseteq \mathcal{X}_c + \text{Im}(B)$ . The solutions to this equation form a subspace very closely related to the constant output subspace discussed in [12]. The constant state subspace is given in terms of the  $A$  and  $B$  matrices below:

$$\mathcal{X}_c = A^\dagger(\text{Im}(A) \cap \text{Im}(B)) + \text{Ker}(A). \quad (6)$$

$A^\dagger$  is the pseudo-inverse of  $A$ . The first term in (6) captures the subspace where control effort can cancel the system dynamics, while the second term characterizes the subspace where there are effectively no dynamics. The GA toolbox for MATLAB is a useful computational tool for carrying out subspace manipulation [13]. The following MATLAB command produces an orthonormal basis for  $\mathcal{X}_c$ : `sums(pinv(A)*ints(A,B),ker(A))`.

One might accidentally mistake the reachable subspace,  $\mathcal{C}$ , for  $\mathcal{X}_c$  or that perhaps  $\mathcal{X}_c$  is a subset of  $\mathcal{C}$  since the equilibrium condition (5) makes it seem that the constant state subspace is related to the reachable subspace. However, they are unrelated in general.

**Remark 1:** *In general the constant state subspace neither contains nor is contained in the reachable subspace:*

(Example with  $\mathcal{C} \subset \mathcal{X}_c$ ) Let  $\text{Rank}(A) = 0$  and let  $\text{Rank}(B) = m < n$ . Since  $\text{Ker}(A) = \mathbb{R}^n$ , the constant-state subspace will be all of  $\mathbb{R}^n$ . However, since  $\text{Rank}(B) = m$  and  $A$  is necessarily  $0_{n \times n}$ , the controllability matrix,  $[BAB^2B \dots A^{n-1}B]$  will have rank  $m$ . Thus,  $\mathcal{C} \subset \mathbb{R}^n = \mathcal{X}_c$ .

(Example with  $\mathcal{X}_c \subset \mathcal{C}$ ) Choose  $A$  to be full rank and  $(A, B)$  controllable so that the controllable/reachable subspace is all of  $\mathbb{R}^n$ . Since  $A$  and  $B$  are each full rank,  $\text{Ker}(A) = 0$ , and  $\dim(A^\dagger(\text{Im}(A) \cap \text{Im}(B))) = \text{Rank}(B) = m$ . If  $m < n$  then the constant-state subspace must be a proper subset of  $\mathbb{R}^n$ ,  $\mathcal{X}_c \subset \mathbb{R}^n = \mathcal{C}$ .

## V. MAIN RESULTS

Next symmetries of the LTI system are discussed and in particular how they are related to the constant state subspace. If  $x(t)$  satisfies (3) with the control  $u_{ol}$ , passing through  $x_0$  at  $t = 0$ , then for any  $v \in \text{Ker}(A)$ ,  $x(t) + v$  is also a solution to (3). This is demonstrated in the following equation:

$$\begin{aligned} x(t) + v &= (x_0 + v) + \int_0^t A(x(\tau) + v) + Bu_{ol} \, d\tau \\ &= \left( x_0 + \int_0^t Ax(\tau) + Bu_{ol} \, d\tau \right) + v. \end{aligned}$$

This follows immediately from the fact that  $v$  is in the null space of  $A$ . One application of this symmetry is that in a stabilizable system a feedback control law which stabilizes the origin can be used to stabilize any  $v \in \text{Ker}(A)$  by a simple change of coordinates. System symmetries have additional applications ranging from motion planning [14] to optimal control [15]. Motivated by the possible applications, it is of interest to use control to artificially introduce additional

symmetries to the system. This symmetry can be extended to the constant state subspace through the appropriate use of control.

**Proposition 1:** If  $x(t)$  is a solution to (3) with control  $u_{ol}$ , then for all  $v \in \mathcal{X}_c$ ,  $x(t) + v$  is a solution with control  $u_{ol} - B^\dagger Av$

*Proof:* Putting the candidate solution into the integral equation yields

$$\begin{aligned} x(t) + v &= (x_0 + v) + \int_0^t A(x(\tau) + v) + B(u_{ol} - B^\dagger Av) \, d\tau \\ &= (x_0 + v) + \int_0^t Ax(\tau) + Av + Bu_{ol} - BB^\dagger Av \, d\tau \end{aligned}$$

Since  $v \in \mathcal{X}_c$ , the control input  $-B^\dagger Av$  pre-multiplied by  $B$  cancels the term  $Av$ . Thus,

$$x(t) + v = \left( x_0 + \int_0^t Ax(\tau) + Bu_{ol} \, d\tau \right) + v. \quad \blacksquare$$

An application of this symmetry is that a motion that is executed in one region of the state space can also be executed in a new location shifted by  $v$ . This may be particularly useful in learning control applications where a motion is rehearsed in an obstacle free region and then executed in a region where performance is critical. Although the symmetry exists for the entire constant state subspace, it is more useful to only consider the intersection of the constant state subspace with the reachable subspace,  $\mathcal{X}_c \cap \mathcal{C}$ , as some of these states may not be stabilizable.

**Proposition 2:** *Suppose  $(A, B)$  is stabilizable. Then a control law exists which stabilizes a fixed state  $x_r$  if and only if  $x_r \in \mathcal{X}_c \cap \mathcal{C}$ .*

*Proof:* ( $\Leftarrow$ ) Suppose  $x_r \in \mathcal{X}_c \cap \mathcal{C}$ . Then there exists a feedback gain  $K$  such that  $(A + BK)$  is Hurwitz. Let  $u = K(x - x_r) + u_r$ , and define the change of coordinate  $\tilde{x} = x - x_r$ . Simplifying state equation in (3) with the change of coordinates yields

$$\dot{\tilde{x}} = (A + BK)\tilde{x} + Ax_r + Bu_r.$$

Since  $x_r \in \mathcal{X}_c$ , there exists a  $u_r$  such that  $0 = Ax_r + Bu_r$ . Choosing such a  $u_r$  leads to an exponentially stable origin in the new coordinates. Hence,  $x_r$  is exponentially stable in the original coordinates.

( $\Rightarrow$ , by contraposition) Suppose now that  $x_r \in (\mathcal{X}_c \cap \mathcal{C})^c$ . Equivalently,  $x_r \in (\mathcal{X}_c)^c \cup (\mathcal{C})^c$ . If  $x_r \in (\mathcal{X}_c)^c$  and at some time  $x(t) = x_r$ , then  $\dot{x}(t) = Ax(t) + Bu(t) \neq 0$  regardless of the choice of control at time  $t$ . It is not possible for  $x_r$  to be an equilibrium and thus is certainly not a stable equilibrium. On the other hand if  $x_r \in (\mathcal{C})^c$ , then  $x_r$  admits a decomposition  $x_r = x_r^{\mathcal{C}} + x_r^{\mathcal{C}^\perp}$  where  $x_r^{\mathcal{C}} \in \mathcal{C}$  and  $x_r^{\mathcal{C}^\perp} \in \mathcal{C}^\perp$ . Since the system is stabilizable, the uncontrollable modes are stable. Hence, any state trajectory  $x(t)$  with similar decomposition into  $\mathcal{C}$ , and  $\mathcal{C}^\perp$  will have  $x^{\mathcal{C}^\perp}(t) \rightarrow 0$  asymptotically. Thus,

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x^{\mathcal{C}}(t) \neq x_r^{\mathcal{C}} + x_r^{\mathcal{C}^\perp},$$

since  $x_r^{c^\perp} \neq 0$ . Thus,  $x_r \notin \mathcal{X}_c \cap \mathcal{C}$  implies that  $x_r$  cannot be stabilized which, by contraposition, completes the proof. ■

#### Related result from geometric control

A related result from the geometric control literature which could be used to come to a conclusion similar to proposition 2 is the output stabilization problem [16]. One can check that states in  $\mathcal{X}_c \cap \mathcal{C}$  are stabilizable by appropriately reformulating the problem as an output stabilization problem and determining the solvability of the problem. This provides a way to verify that a candidate subspace contains states that are indeed stabilizable. However, this approach is limited in that it only verifies that  $\mathcal{X}_c \cap \mathcal{C}$  contains stabilizable states. It does not establish if it contains all stabilizable states.

The following equation outlines how the appropriate output stabilization problem can be formulated. Define an augmented system with matrices  $(\tilde{A}, \tilde{B}, \tilde{D})$  as

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{r} \end{pmatrix} &= \underbrace{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} x \\ r \end{pmatrix} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{\tilde{B}} u, \\ z &= \underbrace{(I - SS^\dagger)}_{\tilde{D}} \begin{pmatrix} x \\ r \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ r(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ x_{ref} \end{pmatrix}. \end{aligned} \quad (7)$$

The added states  $r$  double the dimension of the state space. Let the matrix  $S$  be an orthonormal basis for  $\mathcal{X}_c \cap \mathcal{C}$ . The output regulation problem is to determine if there exists a feedback gain  $F$  such that

$$\lim_{t \rightarrow \infty} \tilde{D}e^{(\tilde{A} + \tilde{B}F)t} \begin{pmatrix} x_0 \\ x_{ref} \end{pmatrix} = 0,$$

which is equivalent to saying the output of the closed loop system will tend to zero. When such a feedback gain exists the problem is said to be solvable. Notice that the augmented system has  $r = x_{ref}$  for all time and if  $x = SS^\dagger r$ , then  $z = 0$ . Further, if  $r \in \mathcal{X}_c \cap \mathcal{C}$  and  $z = 0$ , then  $x = r$  which is the desired result.

To state the necessary and sufficient condition for solvability of the output stabilization problem, as presented in [17], some additional notation is needed. Let  $\alpha(s)$  be the minimal polynomial of the augmented  $\tilde{A}$  matrix in (7). The minimal polynomial can be factored into polynomials with roots in the open left half complex plane and closed right half complex plane,  $\alpha(s) = \alpha^+(s)\alpha^-(s)$ . Now define the subspace  $\mathcal{X}^+(\tilde{A}) = Ker(\alpha^+(\tilde{A}))$  which is the unstable subspace of  $\tilde{A}$ . Next, let  $\mathcal{V}^*$  be the largest  $(\tilde{A}, \tilde{B})$ -controlled invariant subspace contained in  $Ker(\tilde{D})$ . Then the output stabilization problem is solvable if and only if

$$\mathcal{X}^+(\tilde{A}) \subseteq \mathcal{V}^* + \tilde{\mathcal{C}}.$$

$\tilde{\mathcal{C}}$  is the controllable subspace of the augmented system. This condition states in geometric terms that the problem is solvable if and only if the unstable modes of  $\tilde{A}$  are either controllable or can be hidden in the nullspace of  $\tilde{D}$ .

**Corollary 1:** For a system with  $(A, B)$  stabilizable,  $S = \mathcal{X}_c \cap \mathcal{C}$ . The asymptotically reachable subspace is equal to the

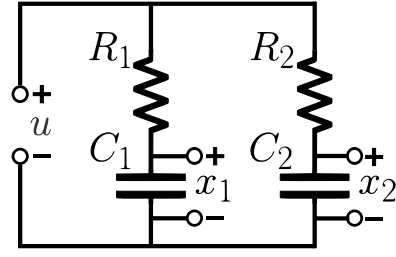


Fig. 3. Two RC filters are connected to a controlled voltage source  $u$ . The states of the system are the voltages across each of the capacitors.

intersection of the constant state subspace with the reachable subspace.

*Proof:*  $((\mathcal{X}_c \cap \mathcal{C}) \subseteq \mathcal{S})$  That  $\mathcal{X}_c \cap \mathcal{C}$  is a subset of the asymptotically reachable subspace is established by constructing the feedback control stabilizing any state in the subspace  $\mathcal{X}_c \cap \mathcal{C}$ . Asymptotic stability of a state is sufficient for it to be asymptotically reachable.

$(\mathcal{S} \subseteq (\mathcal{X}_c \cap \mathcal{C}))$  The reverse inclusion is contained in the reverse direction proof of proposition 2. Recall that for  $x_r \in (\mathcal{X}_c \cap \mathcal{C})^c$ , trajectories cannot reach  $x_r$  asymptotically regardless of the control. Thus,  $x_r \in (\mathcal{X}_c \cap \mathcal{C})^c$  implies that  $x_r \in \mathcal{S}^c$ . Hence, the asymptotically reachable subspace is contained in  $\mathcal{X}_c \cap \mathcal{C}$ . ■

#### Example: RC filter network

This simple example illustrates the distinction between the reachable and asymptotically reachable subspaces as well as provides some intuition. Two RC filters are connected in parallel to a control input voltage as sketched in figure 3. The states  $x_1$  and  $x_2$  are the voltages across the two capacitors. The system dynamics can be obtained using basic circuit analysis and are shown in (8).

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{R_1 C_1} & 0 \\ 0 & \frac{-1}{R_2 C_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{R_1 C_1} \\ \frac{1}{R_2 C_2} \end{pmatrix} u \quad (8)$$

Observe that for  $R_1 C_1 = R_2 C_2$ , the reachable and asymptotically reachable subspaces coincide on  $Span\{(1 \ 1)^T\}$ . In this case the system is not controllable. Controllability is obtained by selecting  $R_1 C_1 \neq R_2 C_2$ . This alteration changes the system from stabilizable to controllable and thus arbitrary voltages can be achieved across either of the capacitors in finite time. However, the asymptotically reachable subspace remains unchanged so that the stabilizable combination of voltages across the capacitors must still lie within  $Span\{(1 \ 1)^T\}$ .

## VI. CONCLUSIONS

The asymptotically reachable subspace and related properties were identified for the  $(A, B)$  pair of a classical LTI system. A stronger condition than the usual reachability, asymptotic reachability identifies the locus of stabilizable reference states as well as being a subspace with useful symmetry.

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