Simultaneous Input and State Estimation of Linear Discrete-Time Stochastic Systems with Input Aggregate Information

Sze Zheng Yong [a] Minghui Zhu [b] Emilio Frazzoli [a]

Abstract— **In this paper, we present filtering algorithms for simultaneous input and state estimation of linear discrete-time stochastic systems when the unknown inputs are partially known, i.e., when some aggregate information of the unknown inputs is available as linear equality or inequality constraints. The stability and optimality properties of the filters are presented and proven using two complementary perspectives. Specifically, we confirm the intuition that the partial input information improves the performance of the filters when a linear input equality constraint is given. On the other hand, given a linear inequality constraint, we show that the estimate error covariance is decreased but the estimates may be biased.**

I. Introduction

The estimation problem for stochastic systems with unknown inputs has applications found across a wide range of disciplines. The unknown disturbance inputs often cannot be modeled by a zero-mean, Gaussian white noise, but are typically not completely unknown, as they may satisfy conservation laws [1] or are bounded by physical laws. For instance, autonomous vehicles do not have knowledge of the control inputs of other vehicles [2] but these inputs are limited by the maximum engine power. Other application examples include real-time estimation of mean areal precipitation during a storm [3], fault detection and diagnosis [4] as well as population and traffic estimation [1], [5], [6].

Literature review. In the extreme scenario where all inputs are observed, the seminal Kalman filter produces statistically optimal estimates of the underlying system states. For the other extreme where the inputs are completely unknown, numerous filters have been recently developed that find minimum-variance unbiased (MVU) estimates of the states only (see, e.g., [3], [7], [8]) or that simultaneously obtain MVU estimates of both states and inputs (see, e.g., [9]–[12]).

Relatively few filters have been developed for the case when the inputs are partially known, although this is a known problem for a long time. Initial research has assumed that these unknown inputs are fixed biases [13] or have known dynamics [5], [14]. More recently, the unknown inputs are assumed to be known at an aggregate level [1], [15] (with equality), which was shown to be suitable for describing conservation laws in a traffic study [6] and aggregated statistics of a population estimation study [5]. However, the proposed filters in [1], [15] assume that the system has no direct feedthrough and only estimate states, although the problem of estimating the partially known inputs is often as important as state estimation. On another hand, to our best knowledge, no filters have been developed for the scenario when the aggregate input is known to satisfy a linear inequality, which, for instance, is the case when the unknown inputs are known to be bounded below (e.g., non-negative) or above (e.g., bounded L_1- and L_∞-norms). Thus, the problem of simultaneously estimating states and inputs in an unbiased minimum-variance manner for systems with linear input aggregate information and without restrictions on the direct feedthrough matrix remains open.

A related set of relevant literature pertains to constrained state estimation, i.e., Kalman filtering with linear or inequality constraints [16], [17]. Of particular interest to the filter development in this paper is the projection method, which exhibits several desirable properties and for which the constrained estimation problem can be either solved in closed form or with a quadratic program.

Contribution. We introduce filtering algorithms for simultaneously estimating both states and unknown inputs when input aggregate information is available in the form of linear equality and inequality constraints. With the input aggregate equality constraint, we can transform the problem via a substitution method into an equivalent problem with no constraints. Hence, the nice properties of the input and state filter developed in an earlier work [12] directly apply, e.g., optimality in a minimum variance unbiased sense, stability and convergence to steady-state. Furthermore, via a "detour" of using a projection method [16] that projects the unconstrained input estimates onto the constraint manifold, we can further show that with the input aggregate information, the mean-squared error of the estimates is decreased.

In contrast, when the input aggregate information is given in the form of a linear inequality, we use the projection method to project the unconstrained input estimates onto the polyhedron described by the linear constraints instead. We show that the error covariance of the input estimates is decreased, but the estimates may become biased, although the bias is imperceptible in our simulation examples.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{C}$ the field of complex numbers. For a vector, $v \in \mathbb{R}^n$, the expectation and L_1-, L_2- and L_∞-norms are denoted by $\mathbb{E}[v]$, $\|v\|_1$, $\|v\|_2$ and $\|v\|_\infty$. The transpose, inverse, Moore-Penrose pseudoinverse, trace and rank of $M \in \mathbb{R}^{p\times q}$ are $M^\top$, M^{-1}, $M^\dagger$, $\mathrm{tr}(M)$ and $\mathrm{rk}(M)$. For a symmetric matrix S, $S \succ 0$ ($S \succeq 0$) indicates S is positive (semi-)definite.

II. Problem Statement and Motivation

Consider the linear time-varying discrete-time system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + G_k d_k + w_k \\ y_k &= C_k x_k + D_k u_k + H_k d_k + v_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector at time k, $u_k \in \mathbb{R}^m$ is a known input vector, $d_k \in \mathbb{R}^p$ is an unknown input vector,

[a] S.Z Yong and E. Frazzoli are with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, USA (e-mail: szyong@mit.edu, frazzoli@mit.edu).

[b] M. Zhu is with the Department of Electrical Engineering, Pennsylvania State University, University Park, PA, USA (e-mail: muz16@psu.edu).

and $y_k \in \mathbb{R}^l$ is the measurement vector. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^l$ are assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matrices, $Q_k = \mathbb{E}[w_k w_k^\top] \succeq 0$ and $R_k = \mathbb{E}[v_k v_k^\top] \succ 0$, respectively. Without loss of generality, we assume that $n \geq l \geq 1$, $l \geq p \geq 0$ and $m \geq 0$, and that the current time variable r is strictly nonnegative. x_0 is assumed to be independent of v_k and w_k for all k. The matrices A_k, B_k, C_k, D_k, G_k and H_k are known, and no assumption about the rank of H_k is made. We also assume that $\max_k(\mathrm{rk}[G_k^\top \ H_k^\top]) = p$ (otherwise, we can retain the linearly independent columns and the "remaining" inputs still affect the system in the same way).

In our previous work [11], [12], we assumed that d_k for all k is completely unknown. In this paper, we consider the case when partial input information is available, i.e., we are given input aggregate information in these linear forms:

1) *Linear equality constraint:*

$$\mathfrak{R}_k d_k = \rho_k, \tag{2}$$

where $\rho_k \in \mathbb{R}^{r_k}$ and $\mathfrak{R}_k \in \mathbb{R}^{r_k \times p}$ are deterministic and known. Furthermore, we assume that $\mathfrak{R}_k$ has full row rank, i.e., $\mathrm{rank}(\mathfrak{R}_k) = r_k$ and $r_k \leq p$ (otherwise, redundant constraints can be removed). This form of aggregate information is found in various contexts (cf. [1]) such as conservation laws, known weighted averages, aggregated statistics, etc. Some concrete examples are when net migration arrivals (input variables) are only known at a national (aggregated) level in the estimation problem of Australian state populations [5] or when the net gain of lane-changing vehicles (unknown inputs) aggregated across all the lanes is equal to zero in a traffic densities study [6].

2) *Linear inequality constraint:*

$$\mathfrak{R}_k d_k \leq \rho_k, \tag{3}$$

where $\rho_k \in \mathbb{R}^{r_k}$ and $\mathfrak{R}_k \in \mathbb{R}^{r_k \times p}$ are deterministic and known (r_k is generally greater than p). This partial information form allows for the incorporation of input bounds (including L_1-norm and L_∞-norm) since unknown disturbance inputs in most practical problems are bounded. For instance, the unknown inputs of other drivers [2] are bounded by the vehicles' engine power, whereas the fault signals in fault detection and diagnosis [4] are oftentimes bounded. In the context of data/signal injection attacks [18], [19] in cyber-physical systems, it may be reasonable to assume that the attacker has limited resources.

The estimator design problem can thus be stated as: *Given a linear discrete-time stochastic system* (1) *with input aggregate information as a linear equality constraint* (2) *and/or a linear inequality constraint* (3)*, design an optimal and stable filter that simultaneously estimates system states and unknown inputs.*

III. Filtering with Partial Input Information

Having motivated the problem at hand, we now proceed to present filtering algorithms for the case when input aggregate information is available, either as an equality or an inequality constraint. For the scenario when *input equality* information (2) is accessible, we consider two equivalent views of the input aggregate information: (i) as information that can be included or substituted into the system equations, or (ii) as a constraint that the input estimate needs to satisfy and also a manifold onto which the estimate can be projected. Fortunately, these two complementary perspectives provide the possibility to study different properties of the filter. Note, however, that the second perspective is solely given for analysis purposes, for reasons that we shall provide at the end of Section III-A.

Algorithm 1 ULISE algorithm (for unconstrained/equality constrained unknown inputs)

Initialize: $P^x_{0|0} = \mathcal{P}^x_0 = (C_{2,0}^\top R_{2,0}^{-1} C_{2,0})^{-1}$; $\hat{x}_{0|0} = \mathbb{E}[x_0] = P^x_{0|0} C_{2,0}^\top R_{2,0}^{-1}(z_{2,0} - D_{2,0}u_0)$; $\hat{A}_0 = A_0 - G_{1,0}\Sigma_0^{-1}C_{1,0}$; $\hat{Q}_0 = G_{1,0}\Sigma_0^{-1}R_{1,0}\Sigma_0^{-1}G_{1,0}^\top + Q_0$; $\hat{d}_{1,0} = \Sigma_0^{-1}(z_{1,0} - C_{1,0}\hat{x}_{0|0} - D_{1,0}u_0)$; $P^d_{1,0} = \Sigma_0^{-1}(C_{1,0}P^x_{0|0}C_{1,0}^\top + R_{1,0})\Sigma_0^{-1}$;

for $k = 1$ to N **do**

▷ Estimation of $d_{2,k-1}$ and d_{k-1}

$\hat{A}_{k-1} = A_{k-1} - G_{1,k-1}M_{1,k-1}C_{1,k-1}$;

$\hat{Q}_{k-1} = G_{1,k-1}M_{1,k-1}R_{1,k-1}M_{1,k-1}^\top G_{1,k-1}^\top + Q_{k-1}$;

$\tilde{P}_k = \hat{A}_{k-1}P^x_{k-1|k-1}\hat{A}_{k-1}^\top + \hat{Q}_{k-1}$;

$\tilde{R}_{2,k} = C_{2,k}\tilde{P}_k C_{2,k}^\top + R_{2,k}$;

$P^d_{2,k-1} = (G_{2,k-1}^\top C_{2,k}^\top \tilde{R}_{2,k}^{-1} C_{2,k} G_{2,k-1})^{-1}$;

$M_{2,k} = P^d_{2,k-1} G_{2,k-1}^\top C_{2,k}^\top \tilde{R}_{2,k}^{-1}$;

$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1} + G_{1,k-1}\hat{d}^u_{1,k-1}$;

$\hat{d}_{2,k-1} = M_{2,k}(z_{2,k} - C_{2,k}\hat{x}_{k|k-1} - D_{2,k}u_k)$;

$\hat{d}_{k-1} = V_{1,k-1}\hat{d}_{1,k-1} + V_{2,k-1}\hat{d}_{2,k-1}$;

$P^d_{12,k-1} = M_{1,k-1}C_{1,k-1}P^x_{k-1|k-1}A_{k-1}^\top C_{2,k}^\top M_{2,k}^\top - P^d_{1,k-1}G_{1,k-1}^\top C_{2,k}^\top M_{2,k}^\top$;

$P^d_{k-1} = V_{k-1}\begin{bmatrix} P^d_{1,k-1} & P^d_{12,k-1} \\ P^{d\top}_{12,k-1} & P^d_{2,k-1}\end{bmatrix} V_{k-1}^\top$;

▷ Time update

$\hat{x}^\star_{k|k} = \hat{x}_{k|k-1} + G_{2,k-1}\hat{d}_{2,k-1}$;

$P^{\star x}_{k|k} = G_{2,k-1}M_{2,k}R_{2,k}M_{2,k}^\top G_{2,k}^\top + (I - G_{2,k-1}M_{2,k}C_{2,k})\tilde{P}_k(I - G_{2,k-1}M_{2,k}C_{2,k})^\top$;

$\tilde{R}^\star_{2,k} = C_{2,k}P^{\star x}_{k|k}C_{2,k}^\top + R_{2,k} - C_{2,k}G_{2,k-1}M_{2,k}R_{2,k} - R_{2,k}M_{2,k}^\top G_{2,k-1}^\top C_{2,k}$;

▷ Measurement update

$\tilde{L}_k = (P^{\star x}_{k|k}C_{2,k}^\top - G_{2,k-1}M_{2,k}R_{2,k})\tilde{R}^{\star\dagger}_{2,k}$;

$\hat{x}_{k|k} = \hat{x}^\star_{k|k} + \tilde{L}_k(z_{2,k} - C_{2,k}\hat{x}^\star_{k|k} - D_{2,k}u_k)$;

$P^x_{k|k} = (I - \tilde{L}_k C_{2,k})G_{2,k-1}M_{2,k}R_{2,k}\tilde{L}_k^\top + \tilde{L}_k R_{2,k}M_{2,k}^\top G_{2,k-1}^\top (I - \tilde{L}_k C_{2,k})^\top + (I - \tilde{L}_k C_{2,k})P^{\star x}_{k|k}(I - \tilde{L}_k C_{2,k})^\top + \tilde{L}_k R_{2,k}\tilde{L}_k^\top$;

▷ Estimation of $d_{1,k}$

$\tilde{R}_{1,k} = C_{1,k}P^x_{k|k}C_{1,k}^\top + R_{1,k}$; $M_{1,k} = \Sigma_k^{-1}$;

$P^d_{1,k} = M_{1,k}\tilde{R}_{1,k}M_{1,k}$;

$\hat{d}_{1,k} = M_{1,k}(z_{1,k} - C_{1,k}\hat{x}_{k|k} - D_{1,k}u_k)$;

end for

In addition, taking the perspective of *input inequality* information (3) as a constraint provides a means to project the input estimates onto the polyhedron described by the linear constraints. However, less can be said about the optimality of the filter in this case. The error covariance of the input estimates decreases with the projection, but the estimate may be biased (though imperceptible in simulations); thus, improvements of the filter remain a subject of future work.

In the following, we denote the error covariances of the constrained (i.e., with input aggregate information) propagated and updated state estimates as $P^{\star x}_{k|k} := \mathrm{Cov}(\tilde{x}^\star_{k|k})$ and

$P^x_{k|k} := \mathrm{Cov}(\tilde{x}_{k|k})$, the error covariances of the constrained input estimate as $P^d_{k-1} := \mathrm{Cov}(\tilde{d}_{k-1})$, and the unconstrained input estimate (i.e., when ignoring the input aggregate information) as $P^{d,u}_{k-1} := \mathrm{Cov}(\tilde{d}^u_{k-1})$.

A. Filtering with Input Aggregate Equality Information

1) Input Equality as Information (via substitution): Viewing the given input aggregate equality (2) as information, we incorporate this partial information via substitution into (1). Note that if $r_k < p$, there exists an infinite number of solutions to (2). Thus, we first obtain all solutions to (2) as

$$d_k = \mathfrak{R}_k^\dagger \rho_k + (I - \mathfrak{R}_k^\dagger \mathfrak{R}_k)\breve{d}_k = \mathfrak{R}_k^\dagger \rho_k + \mathfrak{U}_{1,k}\Xi_k\delta_k, \quad (4)$$

where the latter term represents all possible signals in the null space of $\mathfrak{R}_k$ and we denote with $\mathfrak{R}_k^\dagger := \mathfrak{R}_k^\top(\mathfrak{R}_k\mathfrak{R}_k^\top)^{-1}$ the Moore-Penrose pseudoinverse of $\mathfrak{R}_k$. We have also carried out a singular value decomposition of the matrix $I - \mathfrak{R}_k^\dagger\mathfrak{R}_k$

$$I - \mathfrak{R}_k^\dagger\mathfrak{R}_k := \begin{bmatrix}\mathfrak{U}_{1,k} & \mathfrak{U}_{2,k}\end{bmatrix}\begin{bmatrix}\Xi_k & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathfrak{V}_{1,k}^\top \\ \mathfrak{V}_{2,k}^\top\end{bmatrix} = \mathfrak{U}_{1,k}\Xi_k\mathfrak{V}_{1,k}^\top,$$

and defined $\delta_k := \mathfrak{V}_{1,k}^\top \breve{d}_k \in \mathbb{R}^{\breve{p}_k}$ with $\breve{p}_k = p - r_k$. It can be verified that (4) satisfies (2) for all $\breve{d}_k \in \mathbb{R}^m$.

Then, substituting (4) into the original system (1) and rearranging, we obtain an equivalent system given by

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k' u_k' + G_k'\delta_k + w_k \\ y_k &= C_k x_k + D_k' u_k' + H_k'\delta_k + v_k \end{aligned} \quad (5)$$

where $B_k' := \begin{bmatrix} B_k & G_k\mathfrak{R}_k^\dagger\end{bmatrix}$, $D_k' := \begin{bmatrix} D_k & H_k\mathfrak{R}_k^\dagger\end{bmatrix}$, $G_k' := G_k\mathfrak{U}_{1,k}\Xi_k$, $H_k' := H_k\mathfrak{U}_{1,k}\Xi_k$ and $u_k' = \begin{bmatrix} u_k^\top & \rho_k^\top\end{bmatrix}^\top$, which has the exact form as (1) except with matrices $(A_k, B_k', C_k, D_k', G_k', H_k')$ and with the noise statistics, Q_k and R_k, remaining unchanged. Thus, the ULISE algorithm (cf. Algorithm 1) can be directly applied and by extension, all the nice properties of ULISE such as (i) optimality in the unbiased and minimum variance sense, (ii) the global optimality over the class of all linear state and input estimators, and (iii) stability guarantees and convergence to steady-state (for LTI systems) also hold for the input equality constrained problem. For the sake of brevity, the reader is referred to [12] for proofs and a detailed discussion. Note that some components of δ_k may only be estimated with a delay [12], [20], which implies that some components of d_k may only be estimated with a delay too.

As discussed in [12], [20], strong detectability is a key property for the existence and stability of a simultaneous input and state filter for linear time-invariant systems. Naturally, we would expect that the strong detectability of the original system without input aggregate information (1) would imply that the equivalent system with partial input information (5) is also strongly detectable, whereas the converse may not be true. Thus, the next proposition formalizes the conditions for strong detectability of the equivalent linear time-invariant system with input aggregate information (5), and provides a proof that confirms the intuition.

Proposition 1 (Strong detectability (time-invariant))**.** *Given an input aggregate equality constraint* (2)*, the equivalent system* (5) *is strongly detectability if and only if*

$$\mathrm{rk}\begin{bmatrix} zI - A & -G \\ C & H\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \mathfrak{U}_1\end{bmatrix} = n + \breve{p},\ \forall z \in \mathbb{C}, |z| \geq 1, \quad (6)$$

where $\breve{p} := p - \mathrm{rk}(\mathfrak{R}) =: p - r$. Moreover, strong detectability of the original LTI system (1) *(without input equality information) given by*

$$\mathrm{rk}\begin{bmatrix} zI - A & -G \\ C & H\end{bmatrix} = n + p \ \forall z \in \mathbb{C}, |z| \geq 1, \quad (7)$$

is sufficient but not necessary for the strong detectability condition in (6)*.*

Proof. From the necessary and sufficient condition for the LTI system (A, B', C, D', G', H') given in [12], we have

$$\begin{aligned} \mathrm{rk}\begin{bmatrix} zI - A & -G\mathfrak{U}_1\Xi \\ C & H\mathfrak{U}_1\Xi\end{bmatrix} &= \mathrm{rk}\begin{bmatrix} zI - A & -G \\ C & H\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \mathfrak{U}_1\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \Xi\end{bmatrix} \\ &= \mathrm{rk}\begin{bmatrix} zI - A & -G \\ C & H\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \mathfrak{U}_1\end{bmatrix} = n + \breve{p},\ \forall z \in \mathbb{C}, |z| \geq 1, \end{aligned}$$

where we used the fact that Ξ is invertible. Next, if (7) holds, then $\mathrm{rk}\begin{bmatrix} zI - A & -G\mathfrak{U}_1\Xi \\ C & H\mathfrak{U}_1\Xi\end{bmatrix} = \mathrm{rk}\begin{bmatrix} I & 0 \\ 0 & \mathfrak{U}_1\end{bmatrix} = n + \breve{p}$, i.e., (7) implies (6). The reverse is not necessarily true, as is shown with the following unstable LTI system $A = \begin{bmatrix} 0.1 & 1 \\ 0 & 1.5\end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$ and $H = \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}$, in which (7) does not hold because of an invariant zero at $\{1.5\}$ but with input aggregate information (2) with $\mathfrak{R} = \begin{bmatrix} 1 & 1\end{bmatrix}$, the strong detectability condition (6) holds. ∎

2) Input Equality as Constraint (via projection): If we view the input aggregate information as a constraint, we can further prove that the estimates have better performance (smaller mean-squared errors in the estimates of states and inputs) than without the constraint (similar to state-constrained Kalman filtering [16]), in addition to the properties that are carried over from the properties of ULISE as previously discussed in Section III-A.1. To achieve this, we first show that a commonly employed projection method in [16] leads to estimates in the linear form, which have the desired properties. Then, from the global optimality of the ULISE algorithm over the class of linear estimators [12], we argue that the estimates computed via substitution in Section III-A.1 also have the same properties.

The projection method in [16] is useful for projecting the unconstrained input estimates $\hat{d}^u_k$ onto the constraint surface. Thus, to obtain constrained input estimates, we solve the problem given by

$$\hat{d}_k := \arg\min_{\delta}(\delta - \hat{d}^u_k)^\top \mathcal{W}_k(\delta - \hat{d}^u_k) \text{ s.t. } \mathfrak{R}_k\delta = \rho_k \quad (8)$$

where $\mathcal{W}_k$ is any symmetric positive definite weighting matrix (which we shall see is "optimal" when $\mathcal{W}_k = (P^{d,u}_k)^{-1}$). This constrained optimization problem can be solved analytically, yielding

$$\hat{d}_k = \hat{d}^u_k - \tilde{J}_k(\mathfrak{R}_k\hat{d}^u_k - \rho_k), \quad (9)$$

where $\tilde{J}_k := \mathcal{W}_k^{-1}\mathfrak{R}_k^\top(\mathfrak{R}_k\mathcal{W}_k^{-1}\mathfrak{R}_k^\top)^{-1}$. Moreover, we can find the input estimate error, $\tilde{d}_k := d_k - \hat{d}_k$, from (9) as

$$\begin{aligned} \tilde{d}_k &= d_k - \hat{d}^u_k + \tilde{J}_k(\mathfrak{R}_k\hat{d}^u_k - \rho_k - \mathfrak{R}_k d_k + \mathfrak{R}_k d_k) \\ &= (I - J_k)\tilde{d}^u_k + \tilde{J}_k(\mathfrak{R}_k d_k - \rho_k), \end{aligned} \quad (10)$$

where $\tilde{d}^u_k := d_k - \hat{d}^u_k$ and $J_k := \tilde{J}_k\mathfrak{R}_k$. Note that in this input equality case, $\mathfrak{R}_k d_k - \rho_k = 0$, which further simplifies (10) to $\tilde{d}_k = (I - J_k)\tilde{d}^u_k$, and this is the crux for the derivation of estimate properties, which we state in the following without proof (interested readers are referred to [16, Theorems 1–3]).

Proposition 2. *Let the initial state estimate $\hat{x}_{0|0}$ be unbiased. Then the input estimate is unbiased, i.e., $\mathbb{E}[\hat{d}_{k-1}-d_{k-1}]=0$.*

Proposition 3. *The constrained input estimate $\hat{d}_k$ as given by* (9) *with $\mathcal{W}_k^*=(P_k^{d,u})^{-1}$ has a smaller error covariance than the unconstrained input estimate $\hat{d}_k^u$, i.e., $P_k^d \preceq P_k^{d,u}$.*

Proposition 4. *Among all (symmetric positive definite) weighting matrices, $\mathcal{W}_k$, the estimator of d_k that uses $\mathcal{W}_k^*:=(P_k^{d,u})^{-1}$ has the smallest estimation error covariance P_k^d, i.e., $P_k^d \preceq P_k^{d,\mathcal{W}_k}$.*

A straightforward corollary follows:

Corollary 1. *The constrained and unconstrained input estimate errors also satisfy $\mathrm{trace}(P_k^d) \leq \mathrm{trace}(P_k^{d,u})$ and $\mathrm{trace}(P_k^d) \leq \mathrm{trace}(P_k^{d,\mathcal{W}_k})$ for any $\mathcal{W}_k \succ 0$; and equivalently, $\mathbb{E}[\|\tilde{d}_k\|_2^2] \leq \mathbb{E}[\|\tilde{d}_k^u\|_2^2]$ and $\mathbb{E}[\|\tilde{d}_k\|_2^2] \leq \mathbb{E}[\|\tilde{d}_k^{\mathcal{W}_k}\|_2^2]$ if $\hat{x}_{0|0}$ is unbiased.*

Proof. By Proposition 3, $P_k^d - P_k^{d,u} \preceq 0$. Since the trace of negative semidefinite matrices is non-positive, it follows that $\mathrm{trace}(P_k^d - P_k^{d,u}) \leq 0 \Rightarrow \mathrm{trace}(P_k^d) \leq \mathrm{trace}(P_k^{d,u})$. The latter trace inequality holds similarly from Proposition 4. Finally, the mean-squared error inequalities follows from the equivalence: $\mathrm{tr}(P_k^d)=\mathrm{tr}(\mathbb{E}[\tilde{d}_k\tilde{d}_k^\top])=\mathbb{E}[\|\tilde{d}_k\|_2^2]$. ∎

Therefore, we observe from the above claims that $\mathcal{W}_k=(P_k^{d,u})^{-1}$ is "optimal" in that it minimizes the trace of P_k^d and thus the mean-squared error of the input estimates. Next, we view d_k as the sum of $\hat{d}_k$ and a zero-mean Gaussian noise term, $\tilde{d}_k$. Thus, any additional uncertainty in the input estimate ($\mathrm{trace}(P_k^{d,u}) \geq \mathrm{trace}(P_k^d)$ by Corollary 1) will "increase" the effective noise affecting the system. Intuitively, the larger uncertainty in the noise terms should not lead to a smaller uncertainty in the state estimates, i.e., we expect $\mathrm{trace}(P_k^{x,u}) \geq \mathrm{trace}(P_k^x)$ and $\mathbb{E}[\|\tilde{x}_{k|k}^u\|_2^2] \geq \mathbb{E}[\|\tilde{x}_{k|k}\|_2^2]$, which we observe to hold in simulation and can be explained in steady-state by the following proposition.

Proposition 5 (Steady-state covariance (time-invariant))**.** *Suppose that the filters for the linear time-invariant systems with and without input aggregate equality information are stable. Then, the steady-state error covariance of the state estimates with the input aggregate equality information/constraint, P_∞^x, is smaller than without the input equality, $P_\infty^{x,u}$, i.e., we have $P_\infty^x \preceq P_\infty^{x,u}$ and equivalently, $\mathrm{trace}(P_\infty^x) \leq \mathrm{trace}(P_\infty^{x,u})$ and $\mathbb{E}[\|\tilde{x}_\infty\|_2^2] \leq \mathbb{E}[\|\tilde{x}_\infty^u\|_2^2]$.*

Proof. As discussed above, with the view that $d_k=\hat{d}_k+\tilde{d}_k$, the time-invariant filtering problem is equivalent to

$$\begin{aligned} x_{k+1} &= Ax_k+Bu_k+G\hat{d}_k+Ww_k^{\text{eff}} \\ z_{2,k} &= C_2x_k+D_2u_k+v_{2,k}, \end{aligned} \tag{11}$$

where $\hat{d}_k$ is known, $W=\begin{bmatrix}G & I\end{bmatrix}$, $w_k^{\text{eff}}=\begin{bmatrix}\tilde{d}_k^\top & w_k^\top\end{bmatrix}^\top$ and only the $z_{2,k}$ component of the measurement is used to ensure the unbiasedness of the estimates (cf. [12]). Note that we denote with $\hat{d}_k$ and $\hat{d}_k^u$ (and, $\tilde{d}_k$ and $\tilde{d}_k^u$) the constrained and unconstrained input estimates (and input estimate errors).

By Proposition 3, we have $P_k^d \preceq P_k^{d,u}$, which implies that $P_k^{d,u}=P_k^d+\Delta P_k^d$ for $\Delta P_k^d \succeq 0$, which essentially means that the effective process noise, w_k^{eff}, is larger when no equality input information is incorporated, i.e., $\mathrm{Cov}(w_k^{\text{eff,u}})-\mathrm{Cov}(w_k^{\text{eff}})=:Q_k^{\text{eff,u}}-Q_k^{\text{eff}}=\Delta Q_k$ for some $\Delta Q_k \succeq 0$. By assumption, filters for the linear time-invariant systems with and without input aggregate equality information are stable and hence the steady-state effective process noise covariance $Q_\infty^{\text{eff,u}}=Q_\infty^{\text{eff}}+\Delta Q_\infty$ exists with $\Delta Q_\infty \succeq 0$. Thus, with the same arguments in [21, pp. 312-316] (briefly summarized here due to space constraints), a linearity property of the filters holds in steady-state for any filter gain matrix K, i.e.,

$$\begin{aligned} P_\infty^x &= (I-KC_2)(AP_\infty^xA^\top+WQ_\infty^{\text{eff}}W^\top)(I-KC_2)^\top \\ &\quad +KR_2K^\top \\ \Delta P_\infty^x &= (I-KC_2)(A\Delta P_\infty^xA^\top+W\Delta Q_\infty W^\top)(I-KC_2)^\top \\ P_\infty^{x,u} &= (I-KC_2)(AP_\infty^{x,u}A^\top+WQ_\infty^{\text{eff,u}}W^\top)(I-KC_2)^\top \\ &\quad +KR_2K^\top \end{aligned}$$

where the third equation is a sum of the first two; hence, $P_\infty^{x,u}=P_\infty^x+\Delta P_\infty^x$, where $\Delta P_\infty^x \succeq 0$. It follows that $P_\infty^x \preceq P_\infty^{x,u}$ and equivalently, $\mathrm{trace}(P_\infty^x) \leq \mathrm{trace}(P_\infty^{x,u})$ and $\mathbb{E}[\|\tilde{x}_\infty\|_2^2] \leq \mathbb{E}[\|\tilde{x}_\infty^u\|_2^2]$ (same proof as in Corollary 1). ∎

It is noteworthy that if only the component $z_{2,k} \in \mathbb{R}^{l-p_{H_k}}$ with $p_{H_k}:=\mathrm{rk}(H_k)$ is used in the measurement update in ULISE, instead of $z_{2,k}' \in \mathbb{R}^{l-p_{H_k'}}$ with $p_{H_k'}:=\mathrm{rk}(H_k') \leq p_{H_k}$ (that results from the equivalent system (5)), a slight loss in optimality may be expected. Moreover, it is unclear if the input estimates obtained with the projection method is equivalent to the one with the previous perspective of the input aggregate knowledge as information, which also minimizes the trace of the input error covariance matrix [12].

Nonetheless, since the ULISE algorithm is globally optimal over the class of linear input and state estimators [12], we know that the estimates using the substitution method in Section III-A.1 are no worse than the estimates that result from the projection method, whose estimates are in the linear form (cf. (9)). This means that Propositions 2,3,4,5 and Corollary 1 also hold for the estimates using the method in Section III-A.1. Hence, we recommend the use of ULISE algorithm (see Algorithm 1) with the equivalent system (5) (i.e., to use the substitution method in Section III-A.1) over the projection method in this section. The projection method is provided mainly for analysis purposes and also in preparation for the next section on filtering with input aggregate inequality information.

B. Filtering with Input Aggregate Inequality Information

For the case with input inequality information, the projection method in [16] can be applied to project the unconstrained input estimates $\hat{d}_k^u$ onto the constraint polyhedron, i.e., we now solve the problem given by

$$\hat{d}_k := \arg\min_\delta (\delta-\hat{d}_k^u)^\top \mathcal{W}_k(\delta-\hat{d}_k^u) \text{ s.t. } \mathfrak{R}_k\delta \leq \rho_k \tag{12}$$

where $\mathcal{W}_k$ is any symmetric positive definite weighting matrix. This constrained optimization problem is a quadratic optimization problem and can be efficiently solved using off-the-shelf software packages. From the complementary slackness condition of the Karush-Kuhn-Tucker (KKT) conditions for optimality, we know that we can find the corresponding active constraints (for each i such that $\lambda_i > 0$) from the KKT multiplier vector, λ, that is typically returned by the optimization routine. We denote by $\tilde{\mathfrak{R}}_k$ and $\tilde{\rho}_k$ the rows of $\mathfrak{R}_k$ and elements of ρ_k corresponding to the active constraints; thus, the constrained input estimate $\hat{d}_k$ satisfies $\tilde{\mathfrak{R}}_k\hat{d}_k = \tilde{\rho}_k$. Similar to (9) and (10), we have in this case

$$\begin{aligned} \hat{d}_k &= \hat{d}_k^u - \tilde{J}_k(\tilde{\mathfrak{R}}_k\hat{d}_k^u - \tilde{\rho}_k) \\ \Rightarrow \tilde{d}_k &= (I - J_k)\tilde{d}_k^u + \tilde{J}_k(\tilde{\mathfrak{R}}_k d_k - \tilde{\rho}_k), \end{aligned} \quad (13)$$

where $\tilde{J}_k := \mathcal{W}_k^{-1}\tilde{\mathfrak{R}}_k^\top(\tilde{\mathfrak{R}}_k\mathcal{W}_k^{-1}\tilde{\mathfrak{R}}_k^\top)^{-1}$ and $J_k := \tilde{J}_k\tilde{\mathfrak{R}}_k$. If $\tilde{J}_k(\tilde{\mathfrak{R}}_k d_k - \tilde{\rho}_k) = 0$ (implicitly assumed in [17]), then we have $\tilde{d}_k = (I-J_k)\tilde{d}_k^u$ and thus, Propositions 2, 3 and 4 would directly follow. However, we observed in our simulations that this condition does not generally hold. In this case, we show in the following that the estimates may no longer be unbiased, although the error covariance of the constrained estimates is still lower than that of the unconstrained estimates when d_k is deterministic and $\mathcal{W}_k = (P_k^{d,u})^{-1}$.

Proposition 6. *Suppose* $\tilde{J}_k(\tilde{\mathfrak{R}}_k d_k - \tilde{\rho}_k) \neq 0$ *but* $\mathbb{E}[\tilde{d}_k^u] = 0$ *(a property of ULISE [12]). Then, the inequality constrained input estimate* $\hat{d}_k$ *is biased, i.e.,* $\mathbb{E}[\tilde{d}_k] = \tilde{J}_k(\tilde{\mathfrak{R}}_k d_k - \tilde{\rho}_k) \neq 0$.

Proof. Although $\hat{d}_k^u$ is unbiased [12], since $\tilde{J}_k(\tilde{\mathfrak{R}}_k d_k - \tilde{\rho}_k) \neq 0$, it follows from (13) that input estimate is biased. ■

Proposition 7. *Suppose* d_k *is deterministic. Then, the constrained input estimate* $\hat{d}_k$ *as given by* (12) *with* $\mathcal{W}_k = (P_k^{d,u})^{-1}$ *has a smaller error covariance than the unconstrained input estimate* $\hat{d}_k^u$*, i.e.,* $P_k^d \preceq P_k^{d,u}$*, and equivalently,* $\text{trace}(P_k^d) \leq \text{trace}(P_k^{d,u})$).

Proof. Since d_k is deterministic, from (13), we have

$$\begin{aligned} P_k^d &:= \text{Cov}(\tilde{d}_k) = (I - J_k)\text{Cov}(\tilde{d}_k^u)(I - J_k)^\top \\ &= (I - J_k)P_k^{d,u}(I - J_k)^\top \\ &= P_k^{d,u} - J_k P_k^{d,u} - P_k^{d,u} J_k^\top + J_k P_k^{d,u} J_k^\top \\ &= P_k^{d,u} - J_k P_k^{d,u}, \end{aligned} \quad (14)$$

where the final equality is obtained because it can be verified from the definition of J_k that $P_k^{d,u}J_k^\top = J_k P_k^{d,u} J_k^\top$ with $\mathcal{W}_k = (P_k^{d,u})^{-1}$. Furthermore, $J_k P_k^{d,u} = P_k^{d,u}\tilde{\mathfrak{R}}_k^\top(\tilde{\mathfrak{R}}_k P_k^{d,u}\tilde{\mathfrak{R}}_k^\top)^{-1}\tilde{\mathfrak{R}}_k P_k^{d,u} \succeq 0$ and thus, $\text{trace}(J_k P_k^{d,u}) \geq 0$. It then follows that $P_k^d \preceq P_k^{d,u}$ and $\text{trace}(P_k^d) \leq \text{trace}(P_k^{d,u})$. ■

Proposition 8. *Suppose* d_k *is deterministic. Among all (symmetric positive definite) weighting matrices,* $\mathcal{W}_{k-1}$*, the estimator of* d_k *that uses* $\mathcal{W}_{k-1} = (P_{k-1}^{d,u})^{-1}$ *has the smallest estimation error covariance.*

Proof. Proceeding from (14), the proof is identical to [16, Theorem 3]. ■

Next, with the inequality constrained input estimate $\hat{d}_{k-1}$ given in (12), we proceed as in the ULISE algorithm (cf. Algorithm 1) with the *time update* and *measurement update* steps to find the propagated and updated state estimates:

$$\hat{x}_{k|k}^\star = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1} + G_{k-1}\hat{d}_{k-1} \quad (15)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k}^\star + \tilde{L}_k(z_{2,k} - C_{2,k}\hat{x}_{k|k}^\star - D_{2,k}u_k) \quad (16)$$

where the filter gain $\tilde{L}_k$ can be chosen as in [12] to minimize the state estimate error covariance $\text{trace}(P_{k|k}^x)$:

$$\tilde{L}_k = (P_{k|k}^{\star x}C_{2,k}^\top - \hat{G}_{2,k-1}M_{2,k}R_{2,k})\tilde{R}_{2,k}^{\star\dagger}, \quad (17)$$

where we defined $\hat{G}_{2,k-1} := G_{k-1}(I - \mathfrak{J}_{k-1})V_{2,k-1}$, $\tilde{R}_{2,k}^\star := C_{2,k}P_{k|k}^{\star x}C_{2,k}^\top + R_{2,k} - C_{2,k}\hat{G}_{2,k-1}M_{2,k}R_{2,k} - R_{2,k}M_{2,k}^\top\hat{G}_{2,k-1}^\top C_{2,k}$ and the propagated state estimate error covariance is given by

$$\begin{aligned} P_{k|k}^{\star x} &= \begin{bmatrix} A_{k-1} & G_{k-1} \end{bmatrix} \begin{bmatrix} P_{k-1|k-1}^x & P_{k-1}^{xd} \\ P_{k-1}^{xd\,\top} & P_{k-1}^d \end{bmatrix} \begin{bmatrix} A_{k-1}^\top \\ G_{k-1}^\top \end{bmatrix} + Q_{k-1} \\ &\quad -\hat{G}_{2,k-1}M_{2,k}C_{2,k}Q_{k-1} - Q_{k-1}C_{2,k}^\top M_{2,k}^\top\hat{G}_{2,k-1}^\top, \end{aligned} \quad (18)$$

while the resulting $P_{k|k}^x$ can be computed as

$$\begin{aligned} P_{k|k}^x &= \tilde{L}_k R_{2,k}\tilde{L}_k^\top + (I - \tilde{L}_k C_{2,k})\hat{G}_{2,k-1}M_{2,k}R_{2,k}\tilde{L}_k^\top \\ &\quad + \tilde{L}_k R_{2,k}M_{2,k}^\top\hat{G}_{2,k-1}^\top(I - \tilde{L}_k C_{2,k})^\top \\ &\quad + (I - \tilde{L}_k C_{2,k})P_{k|k}^{\star x}(I - \tilde{L}_k C_{2,k})^\top. \end{aligned} \quad (19)$$

The derivation of the above equations is similar to [12] with the small difference that $\tilde{d}_k$ is as given in (13), where its (unknown) deterministic term does not play a role in the computation of error cross-covariances if we assume that the state estimates remain unbiased. Since this may not hold in general because of the possible bias in the input estimates (Proposition 6), the resulting filter may be suboptimal. Besides, due to the use of only $z_{2,k}$ (cf. discussion at the end of Section III-A.2), the optimality of this filter may further deteriorate. However, this loss of optimality is hardly noticeable in the simulations in Section IV-B, with this filter (summarized in Algorithm 2) still outperforming the case when the input inequality information is ignored.

IV. Illustrative Examples

A. Fault Identification

We consider the benchmark problem of state estimation and fault identification [8], [12] when the system dynamics is plagued by faults, d_k, that influence the system dynamics through the input matrix G_k and the outputs through the feedthrough matrix H_k as well as zero-mean Gaussian white noises. Moreover, we assume partial knowledge of the unknown inputs in the linear form $\mathfrak{R}d_k = \rho_k$.

$$A = \begin{bmatrix} 0.5 & 2 & 0 & 0 & 0 \\ 0 & 0.2 & 1 & 0 & 1 \\ 0 & 0 & 0.3 & 0 & 1 \\ 0 & 0 & 0 & 0.7 & 1 \\ 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}; G = \begin{bmatrix} 1 & 0 & -0.3 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$Q = 10^{-2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}; R = \begin{bmatrix} 1 & 0 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 & 0.3 \\ 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 1 & 0 \\ 0 & 0.3 & 0 & 0 & 1 \end{bmatrix};$$

$$B = 0_{5\times1};\ C = I_5;\ D = 0_{5\times1};\ \mathfrak{R} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The unknown inputs used in this example are

$$d_{k,1} = \begin{cases} 1, & 500 \leq k \leq 700 \\ 0, & \text{otherwise} \end{cases}$$

$$d_{k,2} = \begin{cases} \frac{1}{700}(k - 100), & 100 \leq k \leq 800 \\ 0, & \text{otherwise} \end{cases}$$

$$d_{k,3} = \begin{cases} 3, & 500 \leq k \leq 549, 600 \leq k \leq 649, 700 \leq k \leq 749 \\ -3, & 550 \leq k \leq 599, 650 \leq k \leq 699, 750 \leq k \leq 799 \\ 0, & \text{otherwise.} \end{cases}$$

Figure 1 shows a comparison of the input and state estimates with and without the input aggregate information (i.e., constrained by a linear equality and unconstrained, respectively). As expected, the estimates when given the

Algorithm 2 i-ULISE algorithm (for inequality constrained unknown inputs)

Initialize: $P^x_{0|0} = \mathcal{P}^x_0 = (C_{2,0}^\top R_{2,0}^{-1} C_{2,0})^{-1}$; $\hat{x}_{0|0} = \mathbb{E}[x_0] = P^x_{0|0} C_{2,0}^\top R_{2,0}^{-1}(z_{2,0} - D_{2,0}u_0)$; $\hat{A}_0 = A_0 - G_{1,0}\Sigma_0^{-1}C_{1,0}$; $\hat{Q}_0 = G_{1,0}\Sigma_0^{-1}R_{1,0}\Sigma_0^{-1}G_{1,0}^\top + Q_0$; $\hat{d}^u_{1,0} = \Sigma_0^{-1}(z_{1,0} - C_{1,0}\hat{x}_{0|0} - D_{1,0}u_0)$; $P^d_{1,0} = \Sigma_0^{-1}(C_{1,0}P^x_{0|0}C_{1,0}^\top + R_{1,0})\Sigma_0^{-1}$;

for $k = 1$ to N **do**

▷ Estimation of $d_{2,k-1}$ and d_{k-1}

$\hat{A}_{k-1} = A_{k-1} - G_{1,k-1}M_{1,k-1}C_{1,k-1}$;

$\hat{Q}_{k-1} = G_{1,k-1}M_{1,k-1}R_{1,k-1}M_{1,k-1}^\top G_{1,k-1}^\top + Q_{k-1}$;

$\tilde{P}_k = \hat{A}_{k-1}P^x_{k-1|k-1}\hat{A}_{k-1}^\top + \hat{Q}_{k-1}$;

$\tilde{R}_{2,k} = C_{2,k}\tilde{P}_k C_{2,k}^\top + R_{2,k}$;

$P^{d,u}_{2,k-1} = (G_{2,k-1}^\top C_{2,k}^\top \tilde{R}_{2,k}^{-1} C_{2,k} G_{2,k-1})^{-1}$;

$M_{2,k} = P^{d,u}_{2,k-1} G_{2,k-1}^\top C_{2,k}^\top \tilde{R}_{2,k}^{-1}$;

$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1} + G_{1,k-1}\hat{d}^u_{1,k-1}$;

$\hat{d}^u_{2,k-1} = M_{2,k}(z_{2,k} - C_{2,k}\hat{x}_{k|k-1} - D_{2,k}u_k)$;

$\hat{d}^u_{k-1} = V_{1,k-1}\hat{d}^u_{1,k-1} + V_{2,k-1}\hat{d}^u_{2,k-1}$;

$$P^{d,u}_{12,k-1} = M_{1,k-1}C_{1,k-1}P^x_{k-1|k-1}A_{k-1}^\top C_{2,k}^\top M_{2,k}^\top - P^{d,u}_{1,k-1}G_{1,k-1}^\top C_{2,k}^\top M_{2,k}^\top;$$

$$P^{d,u}_{k-1} = V_{k-1}\begin{bmatrix} P^{d,u}_{1,k-1} & P^{d,u}_{12,k-1} \\ P^{d,u\top}_{12,k-1} & P^{d,u}_{2,k-1}\end{bmatrix} V_{k-1}^\top;$$

$$P^{xd,u}_{2,k-1} = -P^x_{k-1|k-1}A_{k-1}^\top C_{2,k}^\top M_{2,k}^\top - P^{xd,u}_{1,k-1}G_{1,k-1}^\top C_{2,k}^\top M_{2,k}^\top;$$

$P^{xd,u}_{k-1} = P^{xd,u}_{1,k-1}V_{1,k-1}^\top + P^{xd,u}_{2,k-1}V_{2,k-1}^\top$;

$$\hat{d}_{k-1} = \arg\min_\delta (\delta - \hat{d}^u_{k-1})^\top (P^{d,u}_{k-1})^{-1}(\delta - \hat{d}^u_{k-1}) \quad \text{subject to } \mathfrak{R}_{k-1}\delta \le r_{k-1};$$

$\tilde{\mathfrak{R}}_{k-1}$ and $\tilde{r}_{k-1}$ corresponding to active set;

$\mathfrak{J}_{k-1} = P^{d,u}_{k-1}\tilde{\mathfrak{R}}_{k-1}^\top[\tilde{\mathfrak{R}}_{k-1}P^{d,u}_{k-1}\tilde{\mathfrak{R}}_{k-1}^\top]^{-1}\tilde{\mathfrak{R}}_{k-1}$;

$\hat{G}_{2,k-1} = G_{k-1}(I - \mathfrak{J}_{k-1})V_{2,k-1}$;

$P^d_{k-1} = (I - J_{k-1})P^{d,u}_{k-1}(I - J_{k-1})^\top$;

$P^{xd}_{k-1} = P^{xd,u}_{k-1}(I - J_{k-1})^\top$;

▷ Time update

$\hat{x}^\star_{k|k} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1} + G_{k-1}\hat{d}_{k-1}$;

$$P^{\star x}_{k|k} = \begin{bmatrix} A_{k-1} & G_{k-1}\end{bmatrix}\begin{bmatrix} P^x_{k-1|k-1} & P^{xd}_{k-1} \\ P^{xd\,\top}_{k-1} & P^d_{k-1}\end{bmatrix}\begin{bmatrix} A_{k-1}^\top \\ G_{k-1}^\top\end{bmatrix} + Q_{k-1} - \hat{G}_{2,k-1}M_{2,k}C_{2,k}Q_{k-1} - Q_{k-1}C_{2,k}^\top M_{2,k}^\top \hat{G}_{2,k-1}^\top;$$

$$\tilde{R}^\star_{2,k} = C_{2,k}P^{\star x}_{k|k}C_{2,k}^\top + R_{2,k} - C_{2,k}\hat{G}_{2,k-1}M_{2,k}R_{2,k} - R_{2,k}M_{2,k}^\top \hat{G}_{2,k-1}^\top C_{2,k};$$

▷ Measurement update

$\tilde{L}_k = (P^{\star x}_{k|k}C_{2,k}^\top - \hat{G}_{2,k-1}M_{2,k}R_{2,k})\tilde{R}^{\star\dagger}_{2,k}$;

$\hat{x}_{k|k} = \hat{x}^\star_{k|k} + \tilde{L}_k(z_{2,k} - C_{2,k}\hat{x}^\star_{k|k} - D_{2,k}u_k)$;

$$P^x_{k|k} = (I - \tilde{L}_kC_{2,k})\hat{G}_{2,k-1}M_{2,k}R_{2,k}\tilde{L}_k^\top + \tilde{L}_kR_{2,k}M_{2,k}^\top\hat{G}_{2,k-1}^\top(I - \tilde{L}_kC_{2,k})^\top + (I - \tilde{L}_kC_{2,k})P^{\star x}_{k|k}(I - \tilde{L}_kC_{2,k})^\top + \tilde{L}_kR_{2,k}\tilde{L}_k^\top;$$

▷ Estimation of $d_{1,k}$

$\tilde{R}_{1,k} = C_{1,k}P^x_{k|k}C_{1,k}^\top + R_{1,k}$;

$M_{1,k} = \Sigma_k^{-1}$;

$P^{d,u}_{1,k} = M_{1,k}\tilde{R}_{1,k}M_{1,k}$;

$\hat{d}^u_{1,k} = M_{1,k}(z_{1,k} - C_{1,k}\hat{x}_{k|k} - D_{1,k}u_k)$;

end for

input aggregate information are closer to the true states and inputs and have lower error covariances (cf. Figure 2).

B. 3-Area Power System

We consider a power system [19], [22] with three interconnected control areas, each consisting of generators and loads, with tie-lines providing interconnections between areas. A simplified model of the control areas and the tie-lines is given by (cf. parameter definitions in [22, Chap. 10], [23]):

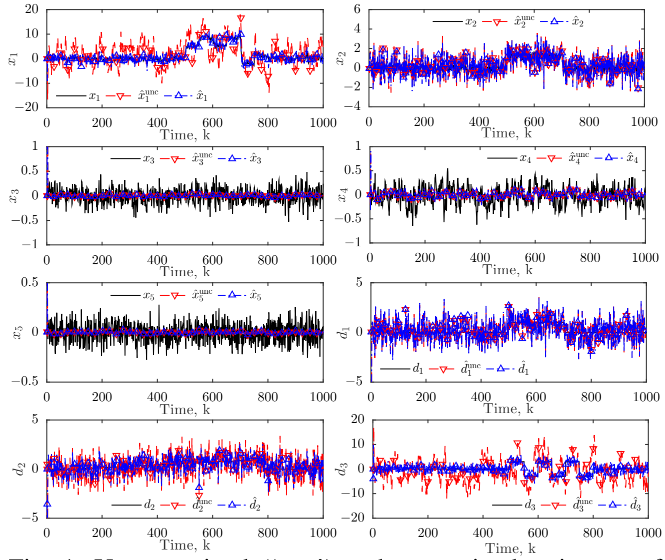

Fig. 1: Unconstrained ('unc') and constrained estimates of states x_1, x_2, x_3, x_4, x_5, and unknown inputs d_1, d_2 and d_3 in Example IV-A.

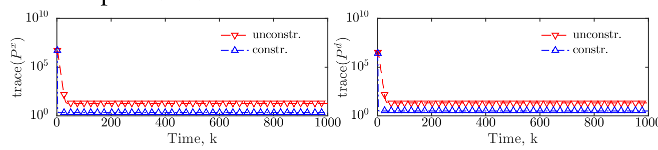

Fig. 2: Traces of unconstrained and constrained estimate error covariance of states, trace(P^x), and unknown inputs, trace(P^d) in Example IV-A.

Control area i: ($i \in \{1, 2, 3\}$)

$$\begin{aligned} \frac{d\Delta\omega_i}{dt} + \frac{D_i\Delta\omega_i}{M_i} - \frac{\Delta P_{mech_i}}{M_i} + \frac{\sum_{j\neq i}\Delta P^{ij}_{tie}}{M_i} &= -\frac{\Delta P_{L_i}}{M_i}, \\ \frac{d\Delta P_{mech_i}}{dt} + \frac{\Delta P_{mech_i}}{T_{CH_i}} - \frac{\Delta P_{v_i}}{T_{CH_i}} &= 0, \\ \frac{d\Delta P_{v_i}}{dt} + \frac{\Delta P_{v_i}}{T_{G_i}} + \frac{\Delta\omega_i}{R_i^f T_{G_i}} &= \frac{\Delta P_{ref_i}}{T_{G_i}}; \end{aligned} \tag{20}$$

Tie-line power flow, P^{ij}_{tie}, between areas i and j:

$$\begin{aligned} \frac{d\Delta P^{ij}_{tie}}{dt} &= T_{ij}(\Delta\omega_i - \Delta\omega_j), \\ \Delta P^{ji}_{tie} &= -\Delta P^{ij}_{tie}, \end{aligned} \tag{21}$$

where $\Delta\omega_i$, ΔP_{mech_i} and $\Delta P v_i$ represent deviations of the angular frequency, mechanical power and steam-valve position from their nominal operating values. We assume that all states are measured (i.e., $C^{q_k}_k = I$) where the system is affected by additive zero mean Gaussian white process and measurement noises with known covariances $Q = 10^{-2} \times \text{diag}(1, 1.6, 2, 1.2, 2.5, 1.4, 0.3, 2.11, 3, 0.2, 0.9, 1.8)$ and $R = 10^{-2} \times \text{diag}(2.1, 0.6, 2.2, 0.2, 1.9, 1.4, 1.3, 1.1, 2.3, 1.2, 0.3, 1.8)$. Moreover, the measurements of $\Delta\omega_i$ are injected with adversarial additive signals $d_k = \begin{bmatrix} d_{k,1} & d_{k,2} & d_{k,3}\end{bmatrix}^\top$, where the attacker is limited in that $\|d_k\|_1 \le 6$, or equivalently, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{bmatrix} d_k \le \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$. The system parameters used in this example are $D_1 = 3$, $R_1^f = 0.03$, $M_1^a = 4$, $T_{CH_1} = 5$, $T_{G_1} = 4$, $D_2 = 0.275$, $R_2^f = 0.07$, $M_2^a = 40$, $T_{CH_2} = 10$, $T_{G_2} = 25$, $D_3 = 2$, $R_3^f = 0.04$, $M_3^a = 35$, $T_{CH_3} = 20$, $T_{G_3} = 15$, $T_{12} = 2.54$, $T_{23} = 1.5$, $T_{31} = 2.5$.

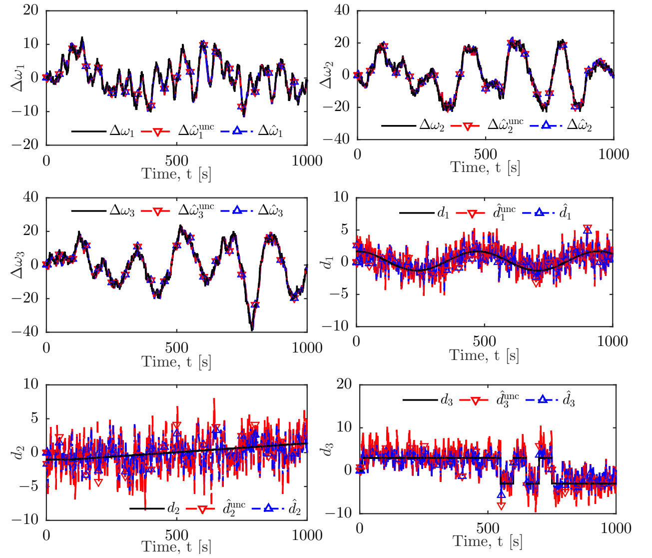

Fig. 3: Unconstrained ('unc') and constrained state and attack magnitude estimates in Example IV-B.

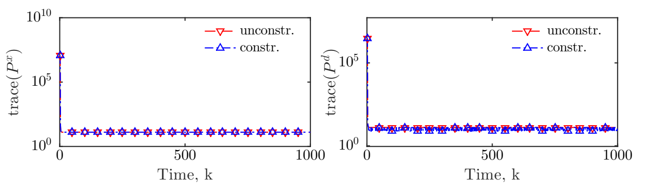

Fig. 4: Traces of unconstrained and constrained estimate error covariance of states, trace(P^x), and unknown inputs, trace(P^d) in Example IV-B.

A comparison of the input and state estimates with and without the linear inequality constraint is shown in Figure 3. We observe little or no improvement to the state estimation, presumably because the attack signals only affect the measurements and not the system dynamics. However, we see that the input estimates are closer to the true inputs and the predicted bias (see Proposition 6) appears insignificant. This can probably be explained by noticing that the bias is introduced only when $\tilde{R}_k \hat{d}_k = \tilde{\rho}_k$ but $\tilde{R}_k d_k \neq \tilde{\rho}_k$. That $\tilde{R}_k \hat{d}_k = \tilde{\rho}_k$ suggests that $\tilde{R}_k d_k - \tilde{\rho}_k$ and thus the bias in (13) is likely to be small. We also observe in Figure 4 that the error covariance of the input is decreased only when the estimates are projected onto the boundary of the polyhedron described by linear inequality constraints, as expected. This also suggests that linear inequality constraints are less informative than equality constraints.

V. Conclusion

We presented filtering algorithms for simultaneous input and state estimation of linear discrete-time stochastic systems for systems in which the inputs are not completely known but some aggregate information of the unknown inputs is available, either as linear equality or inequality constraints. Using two complementary views of the partial input information, we study the properties of the filters including the conditions for stability and optimality in the minimum-variance unbiased sense. We show that the estimate error covariance of the filters decreases when input aggregate information is available in either form. Moreover, given an equality constraint, the estimates remain unbiased but when given as an inequality constraint, the estimates may be biased although the bias is imperceptible in our simulation example.

Acknowledgments

This work was supported by the National Science Foundation, grant #1239182. M. Zhu is partially supported by ARO W911NF-13-1-0421 (MURI) and NSF CNS-1505664.

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