Adaptive Hidden Mode Tracking Control with Input Constraints and Bounded Disturbances

Sze Zheng Yong [a] Emilio Frazzoli [a]

Abstract— **In this paper, we develop an adaptive control approach for hidden mode tracking of uncertain hybrid systems in Brunovsky form that are subject to actuator input amplitude and rate constraints, as well as bounded disturbances. Our approach adapts to the parameters of the hidden mode, and relies on a systematic modification of the reference model to deal with input constraints and disturbances in a stable manner. Global tracking capability is shown for input-to-state stable systems, while for input-to-state unstable systems, the local regions of attraction are characterized. The effectiveness of our input-constrained hidden mode tracking approach is illustrated with a robot walking example.**

I. Introduction

Control systems are typically plagued by uncertainties in the form of disturbance signals and dynamic perturbations. Moreover, control inputs to these systems are constrained in most practical applications due to physical limitations of actuators. It may also be desirable to intentionally impose artificial limits, e.g., to avoid input chattering that can excite unmodeled dynamics, which in turn, may cause plant damage. On the other hand, for hybrid systems (i.e., systems for which the continuous dynamics is described by a finite collection of functions, each corresponding to a *mode*), the mode may be unknown or *hidden* because it is impractical or too costly to measure without interfering with the controlled process or adding unnecessary weight, or when such mode sensors fail. Thus, the tracking control problem of hidden mode hybrid systems (cf. [1], [2] for a detailed model description) with input amplitude and rate constraints can benefit many applications, such as navigation in heterogeneous environments, robotics, manufacturing, electronics, chemical or biological processes, etc.

Literature Review. Control design in the presence of input saturation has been widely studied, especially for known systems with input amplitude limits (see, e.g., [3]–[5] and references therein). The idea of tracking an *adaptive* reference model, i.e., with modifications to the reference model dynamics to deal with control deficiencies due to control amplitude and rate saturation as well as bounded disturbances, has been explored and formally characterized by various authors (e.g., [6]–[13]) for linear time-invariant systems and nonlinear systems in Brunovsky form. This includes a positive (ρ, μ)-modification that is recently proposed in [13] for preventing control amplitude and rate saturation. However, these approaches do not apply to hybrid systems that have switching dynamics and may exhibit state jumps.

On the other hand, tracking controllers for hybrid systems typically assume *full* knowledge and control of the systems' continuous and discrete dynamics [14]–[16]. Since these assumptions are difficult to guarantee due to ever present disturbances and imperfect knowledge of the system parameters and its discrete mode, a hidden mode tracking control design was developed in [2] to adaptively track a well-posed (e.g., with a sufficiently long dwell-time between mode switches) time-varying reference trajectory. The design, however, assumes that there are no constraints on its control inputs.

Contributions. This paper develops an adaptive *hidden mode* tracking control approach for uncertain hybrid systems in Brunovsky form using standard tools of Lyapunov-based control synthesis for hybrid systems when the *input amplitude and rate* are limited. To deal with and prevent input saturation, the reference model is adaptively modified such that in the case when the desired open-loop reference is not feasible due to these saturation limits, the modified reference becomes feasible and remains stable. Moreover, the effects of bounded time-varying disturbances and jump dynamics can be tolerated by further modifying the reference model. Our approach deals with stability issues that arise from combining the hidden mode tracking approach with no input constraints in [2] with the concepts of input-constrained tracking for non-hybrid systems in [13] to *adaptively* track the modified reference trajectory. We provide global and local stability guarantees for open-loop input-to-state stable and unstable systems, respectively, where in the latter case, we provide an estimate of their domains of attraction.

II. Problem Formulation

We consider a class of uncertain hybrid systems $\mathcal{H}$ in Brunovsky form[1] that is assumed to be perturbed by bounded and possibly state-dependent time-varying disturbances, i.e., $|d(\mathbf{x}_p(t), t)| \leq d_{max}$, with known d_{max}:

$$\begin{aligned}(x_p^{(n)}, \dot{q})(t) &= (\mathbf{W}_q^T \mathbf{\Phi}(\mathbf{x}_p(t)) + b_q u(t) + d(\mathbf{x}_p(t), t), 0),\\ &= (f_q(\mathbf{x}_p(t), u(t)), 0), \qquad \mathbf{x}_p(t) \in C_q, \quad (1)\\ (\mathbf{x}_p, q)(t^+) &= (g_q(\mathbf{x}_p(t)), \delta_q(\mathbf{x}_p(t))), \qquad \mathbf{x}_p(t) \in D_q,\end{aligned}$$

where the plant state vector is denoted as $\mathbf{x}_p(t) := [x_p(t), \dot{x}_p(t), \ldots, x_p^{(n-1)}(t)]^T \in \mathbb{R}^n$. For each discrete

[a] S.Z. Yong and E. Frazzoli are with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA, USA (e-mail: szyong@mit.edu, frazzoli@mit.edu).

This work was supported by the National Science Foundation, grant #1239182.

[1]The same approaches developed in this paper can be easily adapted to extend the results to the class of hidden mode hybrid systems with a family of linear time-invariant continuous dynamics (cf. [13]).

state/mode $q \in \mathcal{Q} := \{1, \ldots, |\mathcal{Q}|\}$, $f_q(\mathbf{x}_p(t), u(t))$ is the continuous dynamics, C_q the (closed) flow set, D_q the (closed) jump set, $g_q(\mathbf{x}_p(t))$ the (bounded) discrete transition/reset map and $\delta_q : D_q \to \mathcal{Q}$ the mode transition map. We assume that the reset map $g_q(\mathbf{x}_p(t))$ is Lipschitz continuous for all $q \in \mathcal{Q}$

$$\|g_q(\mathbf{x}_{p,1}(t)) - g_q(\mathbf{x}_{p,2}(t))\| \leq K\|\mathbf{x}_{p,1}(t) - \mathbf{x}_{p,2}(t)\| \quad (2)$$

for some Lipschitz constant $K > 0$. It also follows from (1) that on every open interval on $C_q \setminus D_q$, the mode q remains constant, while the continuous states flow according to $x_p^{(n)}(t) = \mathbf{W}_q^T \mathbf{\Phi}(\mathbf{x}_p(t)) + b_q u(t) + d(\mathbf{x}_p(t), t)$.

The mode is unknown or *hidden* and mode transitions are autonomous, i.e., there is no direct control over the switching mechanism that triggers the discrete events. We also assume that $\mathbf{\Phi}(\mathbf{x}_p(t))$ is a known bounded vector that is independent of the mode q, whereas $\mathbf{W}_q$ and b_q are unknown constant vectors and scalars with known bounds given by $\mathbf{W}_{max} \geq \mathbf{W}_{q,max} \geq \mathbf{W}_q \geq \mathbf{W}_{q,min} \geq \mathbf{W}_{min} > 0$ (element-wise) and $b_{max} \geq b_{q,max} \geq b_q \geq b_{q,min} \geq b_{min} > 0$.

The control input $u(t) \in \mathbb{R}$ is *amplitude and rate limited* with $u_c(t)$ representing the commanded control input before saturation, while u_{max} and $\dot{u}_{max}$ are the actuator amplitude and rate saturation levels. System states $\mathbf{x}_p(t)$ are assumed to be accessible but their derivatives $\dot{\mathbf{x}}_p(t)$ (and specifically the last component) are not[2].

Using the hybrid formalism proposed in [17], solutions ϕ to the hybrid system $\mathcal{H}$ are defined by hybrid arcs on hybrid time domains, which are functions defined on subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ given by the union of intervals of the form $[t_j, t_{j+1}] \times \{j\}$, $t_{j+1} \geq t_j$. Since the mode q remains constant for each j, one can associate each solution of the hybrid system $\mathcal{H}$ with a switching sequence, indexed by an initial state $\phi(0, 0) \in \mathbb{R}^n$:

$$S^{\phi(0,0)} = (t_0, q_0), (t_1, q_1), \ldots, (t_j, q_j), \ldots, (t_N, q_N), \ldots$$

in which the sequence may or may not be infinite. We may take $t_{N+1} = \infty$ in the finite case, with all further definitions and results holding. The corresponding increasing sequence of switching times is denoted as $T_S = t_0, t_1, \ldots, t_j, \ldots, t_N, \ldots$ and the switching modes is denoted as $Q_S = q_0, q_1, \ldots, q_j, \ldots, q_N, \ldots$.

Moreover, if we restrict the solutions to the hybrid system $\mathcal{H}$ to a class of solutions known as *dwell-time* solutions [17], [18] such that the hybrid time domains[3] are given by the union of intervals of the form $[t_j, t_{j+1}] \times \{j\}$, $t_{j+1} \geq t_j + \tau_D$ with dwell-time $\tau_D > 0$, we denote the resulting switching sequence, strictly increasing sequence of switching times and switching modes as $S_{\tau_D}^{\phi(0,0)}$, $T_{S_{\tau_D}}$ and $Q_{S_{\tau_D}}$, respectively.

A. Positive (ρ, μ)-modification

Similar to [8], [9], [13], we consider a control design modification that protects the adaptive input signal from amplitude and rate saturation. This is achieved by defining $u_{max}^{\delta_\mu} := u_{max} - \delta_\mu$, $\dot{u}_{max}^{\delta_\rho} := \dot{u}_{max} - \delta_\rho$, $\Delta u_c(t) := u_{max}^{\delta_\mu}\text{sat}\left(\frac{u_c(t)}{u_{max}^{\delta_\mu}}\right) - u_c(t)$ and $\Delta \dot{u}_{c,\rho}(t) := \dot{u}_{max}^{\delta_\rho}\text{sat}\left(\frac{\dot{u}_{c,\rho}(t)}{\dot{u}_{max}^{\delta_\rho}}\right) - \dot{u}_{c,\rho}(t)$, where δ_μ and δ_ρ are chosen constants that satisfy $0 < \delta_\mu < u_{max}$ and $0 < \delta_\rho < \dot{u}_{max}$. The saturation function is given by

$$\overline{\sigma}\text{sat}\left(\frac{s(t)}{\overline{\sigma}}\right) = \begin{cases} s(t), & |s(t)| \leq \overline{\sigma}, \\ \overline{\sigma}\text{sgn}(s(t)), & |s(t)| > \overline{\sigma}. \end{cases}$$

Using these definitions, we consider a positive (ρ, μ)-modification of the input amplitude and rate using implicit equations given by:

$$u_c(t) := u_d(t) + \mu \Delta u_c(t), \quad (3)$$

$$\dot{u}_{c,\rho}(t) := \dot{u}_{d,\mu}(t) + \rho \Delta \dot{u}_{c,\rho}(t), \quad (4)$$

where $u_d(t)$ is the desired input before μ-modification, $\dot{u}_{d,\mu}(t)$ is the desired input rate after μ-modification but before ρ-modification[4], while $u_c(t)$ and $\dot{u}_{c,\rho}$ are the input amplitude and rate after (ρ, μ)-modification, which will be designed in Section III. Further restrictions on the choice of $\mu > 0$ and $\rho > 0$ will be derived in Theorem 2 for input-to-state unstable systems. The following lemma gives the explicit solutions of $u_c(t)$ and $\dot{u}_{c,\rho}(t)$.

Lemma 1. *For $\mu > 0$ and $\rho > 0$, the explicit solutions to* (3) *and* (4) *$\forall t > 0$ are given by:*

$$u_c(t) = \frac{1}{1+\mu}\left(u_d(t) + \mu u_{max}^{\delta_\mu}\text{sat}\left(\frac{u_d(t)}{u_{max}^{\delta_\mu}}\right)\right), \quad (5)$$

$$\dot{u}_{c,\rho}(t) = \frac{1}{1+\rho}\left(\dot{u}_{d,\mu}(t) + \rho \dot{u}_{max}^{\delta_\rho}\text{sat}\left(\frac{\dot{u}_{d,\mu}(t)}{\dot{u}_{max}^{\delta_\rho}}\right)\right). \quad (6)$$

Proof. The proof for $u_c(t)$ is given in [8], [9], and the same proof applies for $\dot{u}_{c,\rho}(t)$ (cf. [13]). ☐

Remark 1. *As in [13], the input amplitude and rate constraints need not be symmetric. To incorporate asymmetric limits on the control inputs, we can derive the (ρ, μ)-modified command inputs by replacing $\overline{\sigma}\text{sat}\left(\frac{s(t)}{\overline{\sigma}}\right)$ with*

$$\text{asat}(s(t), \underline{\sigma}, \overline{\sigma}) := \begin{cases} s(t), & \underline{\sigma} \leq s(t) \leq \overline{\sigma}, \\ \overline{\sigma}, & s(t) > \overline{\sigma}, \\ \underline{\sigma}, & s(t) < \underline{\sigma}, \end{cases} \quad (7)$$

where $\underline{\sigma}$ represents either u_{min}, $\dot{u}_{min}$, $u_{min}^{\delta_\mu} := u_{min} + \delta_\mu$, or $\dot{u}_{min}^{\delta_\rho} := \dot{u}_{min} + \delta_\rho$; while $\overline{\sigma}$ represents u_{max}, $\dot{u}_{max}$, $u_{max}^{\delta_\mu} := u_{max} - \delta_\mu$ or $\dot{u}_{max}^{\delta_\rho} := \dot{u}_{max} - \delta_\rho$.

B. Reference modification: Impulsive closed-loop higher-order adaptive reference model

Inspired by the approach in [6], [10], a previous work [13] modified the open-loop reference model (ORM) to include control amplitude and rate deficiencies feedback and a tracking error feedback, resulting in a closed-loop higher-order adaptive reference model (CHARM). In this

[2] An extension to the case with accessible $\dot{\mathbf{x}}_p(t)$ is straightforward and a similar idea can be found in [13].

[3] Note that for the sake of conciseness, we may abuse the notation in parts of paper and only specify the continuous time domain, unless the discrete time domain is explicitly needed for clarity.

[4] This definition of the desired input rate is to be distinguished from $\dot{u}_{d,o}$, which is before (ρ, μ)-modification, and $\dot{u}_d$, which is the derivative of u_d after ρ- but before μ-modifications, defined later in Section III.

paper, to cope with impulsive jumps in the system states, we further modify the reference model (i.e., CHARM) to also allow impulsive jumps in the reference model. With this addition, we end up with an impulsive closed-loop higher-order adaptive reference model (iCHARM) of the form:

$$\begin{aligned}\dot{x}_m(t) &= \dot{x}_m^{ORM}(t) + a(\Delta u_d(t)) + c(e(t)),\\ x_m(t^+) &= i_1(x_p(t^+), x_p(t), x_m(t)),\\ \dot{r}(t) &= h(r_d(t), \Delta \dot{u}_d(t)),\\ r(t^+) &= i_2(r(t), x_p(t^+), x_p(t), x_m(t^+), x_m(t)),\end{aligned} \tag{8}$$

where $x_m(t)$ is the model state and $r(t)$ is the reference signal. $\dot{x}_m^{ORM}(t)$ is the open-loop reference model dynamics, which is modified by an adaptive term $a(\Delta u_d(t))$, a tracking error feedback term $c(e(t))$ and a higher order dynamics of $r(t)$ given by $h(r_d(t), \Delta\dot{u}_d(t))$, with $r_d(t)$ being the desired reference signal of the ORM, as well as a novel addition, namely impulsive terms $i_1(\cdot)$ and $i_2(\cdot)$ to "absorb" the state jumps that may lead to violation of the input constraints or to destabilization of the closed-loop system. The tracking error vector and control deficiencies in (8) are defined as

$$e(t) := x_p(t) - x_m(t), \tag{9}$$

$$\Delta u_d(t) := u_{max}\text{sat}\left(\frac{u_c(t)}{u_{max}}\right) - u_d(t), \tag{10}$$

$$\Delta \dot{u}_d(t) := \dot{u}_{max}\text{sat}\left(\frac{\dot{u}_{c,\rho}(t)}{\dot{u}_{max}}\right) - \dot{u}_{d,\mu}(t), \tag{11}$$

with $u_c(t)$ and $\dot{u}_{c,\rho}(t)$ from Lemma 1.

C. Uniform Ultimate Boundedness and Stability

Definition 1 (Uniform Ultimate Boundedness [19])**.** *The solutions of* (1) *are said to be* uniformly ultimately bounded *with ultimate bound β if there exist β and ξ, and for every $0 < \alpha < \xi$, there exists $T = T(\alpha, \beta) \geq 0$ such that*

$$\|\phi(t_0, 0)\| \leq \alpha \Rightarrow \|\phi(t, j)\| \leq \beta, \forall t \geq t_0 + T, (t, j) \in \text{dom}\,\phi.$$

Next, we make use of a weaker version of the stability analysis tool for hybrid systems called multiple Lyapunov functions (MLF) theory [20, Thm. 2.3] as follows:

Proposition 1. *Let $\mathcal{S}$ be the set of all switching sequences associated with the system* (1)*. If for each $S \in \mathcal{S}$, there exist $J \in \mathbb{N}$ and some Lyapunov-like functions $V_q : \mathbb{R}^n \to \mathbb{R}$ for all $q \in \mathcal{Q}$ that satisfy all of the following conditions:*

1) *$V_q(\mathbf{x}_p)$ is positive definite on C_q (i.e., $V_q(\mathbf{x}_p) > 0$ for all $\mathbf{x}_p \neq 0$ and $V_q(0) = 0$),*
2) *$\dot{V}_q(\mathbf{x}_p) \leq 0$ for all $\mathbf{x}_p \in C_q$, and*
3) *$V_{q(t_{j+1})}[j+1] \leq V_{q(t_j)}[j] < \infty$, $\forall j \geq J$ where $V_q[j]$ is defined as the value taken by $V_q(\mathbf{x}_p)$ during the j-th switch-on instant,*

then the system is uniformly ultimately bounded with ultimate bound $\beta := \| \max_{q\in\mathcal{Q}} V_q^{-1}[J]\|$.

Proof. By item 2), the functions $V_q(\mathbf{x}_p)$ are non-increasing during flow. Thus, any increase in $V_q(\mathbf{x}_p)$ is due to state jumps governed by $g_q(\mathbf{x}_p(t))$. But, item 3) essentially imposes an upper bound on $V_q(\mathbf{x}_p)$ after some finite time T despite state jumps. Thus, the proposition follows immediately from Definition 1. □

D. Problem Statement

The problem of an input-constrained tracking control of uncertain hidden mode hybrid systems is relevant for situations when a general desired behavior of the system is given but a specific feasible reference model may be too difficult to design a priori because of input constraints as well as uncertain and complex hybrid system dynamics. Thus, the desired reference model needs to be suitably modified such that when input constraints are imposed, the system remains stable and still exhibits the general desired behavior. On the other hand, when a desired reference model can be tracked without violating input constraints, this desired reference model should be ultimately tracked.

The problem we seek to address is as follows:

Problem 1 (Adaptive Tracking)**.** *Given an open-loop reference model (ORM), design an* adaptive *control signal $u_c(t)$ (or equivalently, $u_d(t)$), as well as the modification terms of iCHARM, i.e., the signals $c(e(t))$, $a(\Delta u_d(t))$, $h(r_d(t), \Delta\dot{u}_d(t))$, $i_1(\cdot)$ and $i_2(\cdot)$ in* (8)*, so that the state $\mathbf{x}_p(t)$ of an uncertain plant with input amplitude and rate constraints tracks the adaptively modified reference model state $\mathbf{x}_m(t)$ with* uniformly ultimately bounded errors*, while all signals of the plant and reference model remain bounded.*

III. Adaptive Tracking Control Design

Our control design addresses Problem 1 and builds on our previous work in [13] and [2], each of which independently tackled a different aspect of Problem 1, namely issues related to input constraints and hidden mode switchings, respectively. However, in an attempt to marry the ideas, many complications arise and in essence, this paper addresses these problems and shows that these ideas can indeed work together with some additional care.

To this end, we shall prove that the tracking error is uniformly ultimately bounded (cf. Definition 1) without violating input constraints, by making use of Proposition 1. In Section III-A, we will deal with challenges associated with impulsive jumps in the system states $\mathbf{x}_p(t)$ and ensure that Conditions 1) and 2) of the proposition is satisfied. On the other hand, in Section III-B, we will handle challenges posed by hidden modes and show how Condition 3) can be satisfied. Then, in Section III-C, we shall provide global and local stability guarantees for open-loop input-to-state stable and unstable systems, respectively, where in the latter case, we provide an estimate of their domains of attraction.

A. Dealing with Input Constraints

Our previous paper [13], which modifies the desired reference model to account for input amplitude and rate constraints, only deals with input constraints for non-hybrid systems. Hence, we need to make sure that impulsive jumps in the system states $\mathbf{x}_p(t)$ according to (1) do not lead to a violation of input constraints or to instability.

Thus, we propose the following modification to the CHARM dynamics in [13] to yield an impulsive version (described in Section II-B) that we call iCHARM:

$$\begin{aligned}
x_m^{(n)}(t) &= \mathbf{k}_x^{*T}\mathbf{x}_m(t) + b_m r(t) + \hat{b}(t)\Delta u_d(t) \\
&\quad + \phi\,\mathrm{sgn}(\mathbf{e}^T(t)P\mathbf{b})d_{max}, \\
\mathbf{x}_m(t^+) &= \mathbf{x}_m(t) + \mathbf{x}_p(t^+) - \mathbf{x}_p(t), \qquad \text{if } \mathbf{x}_p(t^+) \neq \mathbf{x}_p(t), \\
\dot{r}_o(t) &= \dot{r}_d(t) + \Lambda_r(r(t) - r_d(t)), \qquad (12) \\
\dot{r}(t) &= \begin{cases} \dot{r}_o(t) + \frac{\hat{b}(t)}{b_m}\Delta\dot{u}_d, & |u_c(t)| \leq u_{max}^{\delta_\mu}, \\ \dot{r}_o(t) + \frac{(1+\mu)\hat{b}(t)}{b_m}\Delta\dot{u}_d, & u_{max}^{\delta_\mu} < |u_c(t)| \leq u_{max}, \\ \dot{r}_o(t), & \text{otherwise}, \end{cases} \\
r(t^+) &= r(t) + \frac{1}{b_m}(\hat{\mathbf{W}}(t)^T(\mathbf{\Phi}(\mathbf{x}_p(t^+)) - \mathbf{\Phi}(\mathbf{x}_p(t))) \\
&\quad - \mathbf{k}_x^{*T}(\mathbf{x}_p(t^+) - \mathbf{x}_p(t))), \qquad \text{if } \mathbf{x}_p(t^+) \neq \mathbf{x}_p(t),
\end{aligned}$$

with $\mathbf{b} = \begin{bmatrix}0 & \ldots & 0 & 1\end{bmatrix}^T$, and any $\phi \geq 1$ and $\Lambda_r < 0$ (see additional constraint on Λ_r in Theorem 2 for input-to-state unstable systems). $\dot{r}(t)$ is right-continuous, $\mathbf{e}(t) := \mathbf{x}_p(t) - \mathbf{x}_m(t)$ is the tracking error vector, $\mathbf{x}_m(t) := [x_m(t), \dot{x}_m(t), \ldots, x_m^{(n-1)}(t)]^T$ is the model state vector and b_m is the model gain parameter, while $\Delta u_d(t)$ and $\Delta\dot{u}_d(t)$ are given by (10) and (11). $\mathbf{k}_x^*$ is chosen such that $A = \begin{bmatrix} 0 & I \\ \mathbf{k}_x^{*T} & \end{bmatrix}$ is Hurwitz (such that the $\mathbf{x}_m$ dynamics remain stable), while $P = P^T$ is the solution of the algebraic Lyapunov equation $A^TP + PA = -Q$ for arbitrary $Q \succ 0$.

A noteworthy addition to the CHARM dynamics in [13] is the jumps in $\mathbf{x}_m$ and r in (12), which absorb the state jumps that would otherwise appear in the control input and violate input constraints. Note also that the case $\mathbf{x}_p(t^+) \neq \mathbf{x}_p(t)$ above is not required for switched systems (i.e., when there are no state jumps).

We will now show that the iCHARM dynamics above is compatible with the Lyapunov-based adaptive control approach that we will take. To this end, we first define a control Lyapunov function candidate for each mode q:

$$\begin{aligned}
V_q(\mathbf{x}_p(t)) &= \mathbf{e}(t)^TP\mathbf{e}(t) + \tilde{\mathbf{W}}_q(t)^T\Gamma_W^{-1}\tilde{\mathbf{W}}_q(t) \\
&\quad + \gamma_b^{-1}\tilde{b}_q(t)^2 + \tilde{u}(t)^2, \qquad (13)
\end{aligned}$$

where $\hat{b}(t)$ and $\hat{\mathbf{W}}(t)$ are estimates of b_q and $\mathbf{W}_q$, whereas the parameter errors are $\tilde{b}_q(t) = \hat{b}(t) - b_q$ and $\tilde{\mathbf{W}}_q(t) = \hat{\mathbf{W}}(t) - \mathbf{W}_q$ for each mode $q \in \mathcal{Q}$, and the tracking error vector $\mathbf{e}$ is as defined above. Moreover, we defined the "desired" input (as if $\dot{\mathbf{x}}_p(t)$ were accessible) as $u_d^\star(t) = \frac{(\mathbf{k}_x^{*T}\mathbf{x}_p(t) + b_m r(t) - \hat{\mathbf{W}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)))}{\hat{b}(t)}$, and the input error as $\tilde{u}(t) := u_d(t) - u_d^\star(t)$.

To satisfy Condition 2) of Proposition 1, we choose the following control and adaptation laws:

Control Law:

$$\begin{aligned}
\dot{u}_{d,o}(t) &= -k_{\tilde{u}}\tilde{u}(t) - \mathbf{e}(t)^TP\mathbf{b}\hat{b}(t) + \frac{1}{\hat{b}(t)}\big(b_m\dot{r}_o(t) \\
&\quad - \dot{\hat{b}}(t)u_d^\star(t) - \dot{\hat{\mathbf{W}}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)) \\
&\quad + (\mathbf{k}_x^{*T} - \hat{\mathbf{W}}(t)^T\Phi'(\mathbf{x}_p(t))) \\
&\quad \begin{bmatrix} \begin{bmatrix}0_{n-1} & I_{n-1}\end{bmatrix}\mathbf{x}_p(t) \\ \hat{\mathbf{W}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)) + \hat{b}(t)u(t) \\ -\varphi\,\mathrm{sgn}(\tilde{u}(t)L(t))d_{max} \end{bmatrix}\big),
\end{aligned}$$

$$\dot{u}_{d,\mu}(t) = \begin{cases} \dot{u}_{d,o}(t), & |u_c(t)| \leq u_{max}^{\delta_\mu}, \\ \frac{1}{1+\mu}\dot{u}_{d,o}(t), & u_{max}^{\delta_\mu} < |u_c(t)| \leq u_{max}, \\ 0, & \text{otherwise}, \end{cases} \qquad (14)$$

$$\begin{aligned}
\dot{u}_d(t) &= -k_{\tilde{u}}\tilde{u}(t) - \mathbf{e}(t)^TP\mathbf{b}\hat{b}(t) + \frac{1}{\hat{b}(t)}\big(b_m\dot{r}(t) \\
&\quad - \dot{\hat{b}}(t)u_d^\star(t) - \dot{\hat{\mathbf{W}}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)) \\
&\quad + (\mathbf{k}_x^{*T} - \hat{\mathbf{W}}(t)^T\Phi'(\mathbf{x}_p(t))) \qquad (15) \\
&\quad \begin{bmatrix} \begin{bmatrix}0_{n-1} & I_{n-1}\end{bmatrix}\mathbf{x}_p(t) \\ \hat{\mathbf{W}}(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)) + \hat{b}(t)u(t) \\ -\varphi\,\mathrm{sgn}(\tilde{u}(t)L(t))d_{max} \end{bmatrix}\big),
\end{aligned}$$

Adaptation Law:

$$\begin{aligned}
\dot{\hat{\mathbf{W}}}(t) &= \Gamma_W\mathbf{\Phi}(\mathbf{x}_p(t))\left(\mathbf{e}(t)^TP\mathbf{b} - \frac{L(t)\tilde{u}(t)}{\hat{b}(t)}\right), \\
\dot{\hat{b}}_o(t) &= \gamma_b u_{max}\mathrm{sat}\left(\frac{u_c(t)}{u_{max}}\right)\left(\mathbf{e}(t)^TP\mathbf{b} - \frac{L(t)\tilde{u}(t)}{\hat{b}(t)}\right), \\
\dot{\hat{b}}(t) &= \begin{cases} 0, & \hat{b}(t) \leq b_{min} \wedge \dot{\hat{b}}_o(t) < 0, \\ \dot{\hat{b}}_o(t), & \text{otherwise}, \end{cases} \qquad (16)
\end{aligned}$$

for any $\varphi \geq 1$, $k_{\tilde{u}} > 0$, $\Gamma_W = \Gamma_W^T \succ 0$ and $\gamma_b > 0$. $L(t) = (\mathbf{k}_x^{*T} - \hat{\mathbf{W}}(t)^T\mathbf{\Phi}'(\mathbf{x}_p(t)))\begin{bmatrix}0 & \ldots & 0 & 1\end{bmatrix}^T$ is the last element of $\mathbf{k}_x^{*T} - \hat{\mathbf{W}}(t)^T\mathbf{\Phi}'(\mathbf{x}_p(t))$ and $\mathbf{\Phi}'(\mathbf{x}_p(t))$ is the Jacobian matrix of $\mathbf{\Phi}(\mathbf{x}_p(t))$. Note that $u_d(t)$ is obtained from integrating (15) with $u_d(0) = u_d^\star(0)$, while $u_c(t)$, $\dot{u}_{c,\rho}(t)$, $\Delta u_d(t)$ and $\Delta\dot{u}_d(t)$ are as given in (5), (6), (10) and (11).

Lemma 2. *The control and adaptation laws in* (14)*,* (15) *and* (16) *satisfy Conditions 1) and 2) of Proposition 1 for the control Lyapunov-like function given in* (13)*.*

Proof. With the control and adaptation laws, the tracking error and input error dynamics are given by

$$\begin{aligned}
\dot{\mathbf{e}}(t) &= A\mathbf{e}(t) - \mathbf{b}(\tilde{\mathbf{W}}_q(t)^T\mathbf{\Phi}(\mathbf{x}_p(t)) + \tilde{b}_q(t)u(t) \\
&\quad - d(\mathbf{x}_p(t), t) + \phi\,\mathrm{sgn}(\mathbf{e}(t)^TP\mathbf{b})d_{max} \qquad (17) \\
&\quad - \hat{b}(t)\tilde{u}(t)), \\
\dot{\tilde{u}}(t) &= \frac{1}{\hat{b}}L(t)(\tilde{\mathbf{W}}_q(t)^T\mathbf{\Phi} + \tilde{b}_q(t)u(t) \\
&\quad - \varphi\,\mathrm{sgn}(\tilde{u}(t)L(t))d_{max} - d(\mathbf{x}_p(t), t)) \qquad (18) \\
&\quad - k_{\tilde{u}}\tilde{u}(t) - \mathbf{e}(t)^TP\mathbf{b}\hat{b}(t).
\end{aligned}$$

It is then straightforward to show that the $V_q(\mathbf{x}_p(t)) > 0$ in (13) has a negative semidefinite time derivative:

$$\dot{V}_q(\mathbf{x}_p(t)) \leq -\mathbf{e}(t)^TQ\mathbf{e}(t) - 2k_{\tilde{u}}\tilde{u}(t)^2 \leq 0, \qquad (19)$$

as required by Conditions 1) and 2) of Proposition 1. ☐

B. Dealing with Hidden Modes

Motivated by a previous work [2], we will assume that mode switchings are sufficiently infrequent, i.e., there exists a dwell-time τ_D between switches, to cope with hidden modes of the hybrid system. Specifically, we assume the existence of a dwell-time solution that satisfies Condition 3) of Proposition 1.

Lemma 3 (Dwell-time requirement I)**.** *Suppose the switching sequence $S_{\tau_D}^{\phi(t_J,J)}$ starting after some finite time (t_J, J) has a dwell-time τ_D given by*

$$\tau_D \geq \tau_D^V \geq \tau_{D,j}^V$$

for all $j \geq J$ with

$$\tau_{D,j}^V = \frac{\lambda_{max}(P)}{\lambda_{min}(Q)} \ln \left| \frac{V_{q_j}(\mathbf{x}_p(t_j)) - \Delta V_{max}}{V_{q_j}(\mathbf{x}_p(t_j)) - \Delta V_{max} - \Delta V_{j+1}} \right|, \tag{20}$$

with the following definitions: $\Delta V_{j+1} := [V_{q_{j+1}}(g_{q_j}(\mathbf{x}_p, u)) - V_{q_j}(\mathbf{x}_p)](t_{j+1})$, and $\Delta V_{max} := \max_{t \in \mathcal{I}(j,j+1)} \tilde{\mathbf{W}}_{q_j}(t)^T \Gamma_W^{-1} \tilde{\mathbf{W}}_{q_j}(t) + \gamma_b^{-1} \tilde{b}_{q_j}(t)^2 + \tilde{u}(t)^2$ where $\mathcal{I}(j, j+1)$ is the hybrid time interval between jumps. Then, Condition 3) of Proposition 1 holds.

Proof. For the sake of brevity, only a proof sketch will be given. From (19), using an approach similar to [21, pp. 91-93], we can find after some lengthy simplifications that for each hybrid time interval between jumps, $\mathcal{I}(j, j+1)$,

$$\dot{V}_q(\mathbf{x}_p) \leq \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} (V_q(\mathbf{x}_p) - \Delta V_{max}). \tag{21}$$

In order for Condition 3) of Proposition 1 to hold, over the hybrid time interval between jumps $\mathcal{I}(j, j+1)$ before the next jump, the value of $V_q(\mathbf{x}_p)$ must decrease from $V_q(\mathbf{x}_p(t_j))$ by at least the increase ΔV_{j+1} that is caused by the jump $g_{q_j}(\mathbf{x}_p, u)$. Thus, by applying the Comparison Lemma [19] to (21), it can be verified that the minimum dwell-time for all $j \geq J$ is given by (20). The maximum dwell-time across all $j \geq J$ is then denoted τ_D^V. □

C. Global and Local Stability Guarantees

To find global and local stability guarantees for system (1), we first note that the tracking errors are uniformly ultimately bounded in the next lemma that follows directly from Proposition 1 since Conditions 1), 2) and 3) hold.

Lemma 4 (Boundedness of tracking error)**.** *For a given reference trajectory and impulse dynamics that satisfy Lemma 3, the control and adaptation laws in* (14)*,* (15) *and* (16)*, as well as the iCHARM dynamics in* (12) *guarantee that the solution to the tracking error dynamics $\mathbf{e}(t)$ is uniformly ultimately bounded, i.e., $\|\mathbf{e}(t)\| < e_{max}$ for all $t \geq t_0 + T$ for some T. Moreover, the input and parameter 'errors' are also ultimately bounded, i.e., $|\tilde{u}(t)| < \tilde{u}_{max}$, $|\tilde{b}_q(t)| < \tilde{b}_{max}$ and $\|\tilde{\mathbf{W}}_q(t)\| < \tilde{W}_{max}$ for all $t \geq t_0 + T$ for some T.*

Nonetheless, it is not sufficient that the tracking errors $\mathbf{e}(t) = \mathbf{x}_p(t) - \mathbf{x}_m(t)$ are bounded. In fact, it would defeat the control purpose unless we additionally show the boundedness of either the plant or model states, i.e., $\mathbf{x}_p(t)$ or $\mathbf{x}_m(t)$, and thus the boundedness of both. Clearly, for systems (1) with input-to-state *stable* continuous dynamics, the plant states (and hence the model states) can be shown to be globally bounded since both input $u(t)$ and disturbance $d(\mathbf{x}_p(t), t)$ are bounded, and the reset map $g_q(\mathbf{x}_p(t))$ is bounded by assumption. Thus, the following theorem follows directly.

Theorem 1 (Global stability for systems with input-to-state stable continuous dynamics)**.** *Suppose the resulting input-to-state stable plant trajectory* (1) *with the iCHARM dynamics in* (12) *has a dwell-time $\tau_D \geq \tau_D^V$, where τ_D^V is the dwell-time given in Lemma 3, then the plant states (and hence the model states) are* globally *uniformly ultimately bounded.*

For the case when the continuous dynamics of system (1) is input-to-state *unstable*, global uniform ultimate boundedness of the plant states is not guaranteed. In this case, we instead provide in Theorem 2 the characterization of its local domain of attraction.

For analysis purposes, we assume that in the operating region of interest, $||\mathbf{x}_p(t^+) - \mathbf{x}_p(t)|| \leq J_{max}$ and $||\mathbf{\Phi}(\mathbf{x}_p(t^+)) - \mathbf{\Phi}(\mathbf{x}_p(t))|| \leq J_{max}^{\Phi}$. We also assume that the control input authority is greater than the disturbance input, i.e., there exists $R > 0$ such that $\mathbf{x}_p(t) \in \mathcal{B}_R := \{\mathbf{x}_p(t) : \|\mathbf{x}_p(t)\| \leq R\}$ and $b_{min} u_{max} \geq \max_{\mathbf{x}_p(t) \in \mathcal{B}_R} \bar{d}_q(\mathbf{x}_p(t), t)$, where $\bar{d}_q(\mathbf{x}_p(t), t) := |\mathbf{W}_q^T \mathbf{\Phi}(\mathbf{x}_p(t)) + d(\mathbf{x}_p(t), t)| \leq |\mathbf{W}_q^T \mathbf{\Phi}(\mathbf{x}_p(t))| + d_{max} \leq \bar{d}_{max}$ for all q. We further assume that $\|\mathbf{W}_q\| \leq W_{max}$, $\|\dot{\mathbf{\Phi}}(\mathbf{x}_p(t))\| \leq \dot{\Phi}_{max}$ and that the upper bounds are known. Thus, $|\mathbf{W}_q^T \mathbf{\Phi}(\mathbf{x}_p(t))| \leq W_{\Phi}^{max} := \bar{d}_{max} - d_{max}$ and $\exists \psi$ such that $\tilde{W}_{max} + W_{max} = \psi \tilde{b}_{max}$.

Before moving forward, we would like to note that an additional dwell-time τ_D^r is required for input-to-state unstable systems. This is because jumps in $r(t)$ may cause the system state to exit the domain of attraction. Thus, we require a 'second' dwell-time τ_D^r such that the signal r at the end of the each interval is small enough, i.e.,

$$\begin{aligned} r[j] \leq r_{max}^{\mathrm{I}} &:= r_{max,B}^{\mathrm{I}} \\ &\quad - \frac{||\mathbf{k}_x^*||J_{max} + (W_{max} + \tilde{W}_{max}) J_{max}^{\Phi}}{b_m}, \\ r_{max,B}^{\mathrm{I}} &< \frac{b_{min} \lambda_{min}(Q) \eta}{2\kappa b_m b_q} - \bar{d}_{max} - \frac{b_{min}}{b_m} \tilde{u}_{max}, \end{aligned} \tag{22}$$

where $r[j]$ denotes the value of $r(t)$ at the end of the hybrid time-interval $\mathcal{I}(j-1, j)$, $\eta := \frac{2|b_q u_{max} - \bar{d}_{max}|}{|\lambda_{min}(Q) - 2\|P\mathbf{b}\|\|\mathbf{k}_x^*\||}$ and $\kappa := \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$, and the upper bound on $r_{max,B}^{\mathrm{I}}$ is the bound given in [13, Theorem 5] for nonlinear systems in Brunovsky form without jumps. Note that the latter term in r_{max}^{I} is crucial because it ensures that even with jumps in r, it still results in $r(t^+) \leq r_{max,B}^{\mathrm{I}}$. The next lemma provides this 'second' dwell-time, and can proven in the same manner as Lemma 3; thus the proof is omitted for brevity.

Lemma 5 (Dwell-time requirement II)**.** *Let the switching sequence $S_{\tau_D}^{\phi(t_J,J)}$ starting after some finite time (t_J, J) have a dwell-time given by*

$$\tau_D \geq \tau_D^r \geq \tau_{D,j}^r$$

for all $j \geq J$ with

$$\tau_{D,j}^r = \frac{1}{\Lambda_r} \ln \left| \frac{r(t^+) - \Delta r_{max}}{r(t^-) - \Delta r_{max}} \right|, \tag{23}$$

with $\Delta r_{max} := \max_{t \in \mathcal{I}(j,j+1)} (\dot{r}(t) - \dot{r}_o(t))$, where $\mathcal{I}(j, j+1)$ is the hybrid time interval between jumps, while t^+ and t^- are the times before and after the j-th jump (at the start *of interval $\mathcal{I}(j, j+1)$). Then, $r(t^+_{j+1}) \leq r^{\mathrm{I}}_{max,B}$ holds for all $j \geq J$.*

With the above, we now provide the characterization of the local domain of attraction for systems (1) with input-to-state unstable continuous dynamics.

Theorem 2 (Local stability for systems with input-to-state unstable continuous dynamics)**.** *Suppose the resulting input-to-state unstable plant trajectory* (1) *with the iCHARM dynamics in* (12) *has a dwell-time*

$$\tau_D \geq \max\{\tau^V_D, \tau^r_D\},$$

where τ^V_D and τ^r_D are the dwell-times given in Lemmas 3 and 5, and let the minimum and maximum of the desired reference signal be such that

$$-r^{\mathrm{I}}_{max} < r^{min}_d \leq r_d(t) \leq r^{max}_d < r^{\mathrm{I}}_{max}$$

with r^{I}_{max} given in (22)*. For given lower and upper bounds on $\dot{r}_d(t)$ such that*

$$\dot{r}^{min}_d \leq \dot{r}_d(t) \leq \dot{r}^{max}_d,$$

let the design parameter Λ_r be chosen to satisfy

$$\Lambda_r \leq -\frac{2D^{\mathrm{I}} + \dot{r}^{max}_d - \dot{r}^{min}_d}{2r^{\mathrm{I}}_{max} - (r^{min}_d - r^{max}_d)},$$

and for arbitrary $0 < \delta_\mu < u_{max}$ and $0 < \delta_\rho < \dot{u}_{max}$, let the design parameters μ and ρ be selected such that:

$$\mu > \max\{\frac{\|\mathbf{k}^*_x\|\|P\mathbf{b}\|\eta + b_m r^{\mathrm{I}}_{max} + W^{max}_\Phi}{b_{min}\delta_\mu} + \frac{\tilde{u}_{max} + u_{max}}{\delta_\mu} - 2, 0\},$$

$$\rho > \max\{\frac{1}{\delta_\rho}(\dot{u}_{max} + C^{\mathrm{I}} + \frac{b_m}{b_{min}}|\Lambda_r|(r^{\mathrm{I}}_{max} + r^{max}_d)) - 2, 0\},$$

where $|\dot{u}_{d,\mu}(t)| \leq C^{\mathrm{I}} + \frac{b_m}{b_{min}}|\Lambda_r|(r^{\mathrm{I}}_{max} + r^{max}_d)$, with

$$\begin{aligned} C^{\mathrm{I}} := \frac{1}{b_{min}}\big[&(\|\mathbf{k}^*_x\| + (W_{max} + \tilde{W}_{max})\Phi'_{max})(\varphi d_{max} \\ &+ \sqrt{\lambda_{min}(P)}\eta\|P\mathbf{b}\| + (W_{max} + \tilde{W}_{max})\Phi_{max} \\ &+ (\tilde{b}_{max} + b_q)u_{max}) + \|\Gamma_W\|\Phi^2_{max}e_{max}\|P\mathbf{b}\| \\ &+ e_{max}\|P\mathbf{b}\|(b_q + \tilde{b}_{max}) + k_{\tilde{u}}\tilde{u}_{max}\big] + b_m\dot{r}^{max}_d \\ &+ \frac{\gamma_b u_{max} e_{max}\|P\mathbf{b}\|}{b^2_{min}}\big[\|\mathbf{k}^*_x\|\sqrt{\lambda_{min}(P)}\eta\|P\mathbf{b}\| \\ &+ b_m r^{\mathrm{I}}_{max} + (W_{max} + \tilde{W}_{max})\Phi_{max}\big], \\ D^{\mathrm{I}} := \frac{(1+\mu)b_{max}}{b_m}&(\dot{u}_{max} + C^{\mathrm{I}}). \end{aligned}$$

If the system initial condition and the initial value of the candidate Lyapunov function satisfy

- $\mathbf{x}_p(0)^T P \mathbf{x}_p(0) < \lambda_{min}(P)\eta^2\|P\mathbf{b}\|^2$,
- $\sqrt{V(\mathbf{x}_p(0))} < \sqrt{\frac{1}{\gamma_b}}\left(\frac{\lambda_{min}(Q) - 2\kappa\frac{b_m b_q}{b_{min}}\frac{r^{\mathrm{I}}_{max} + \bar{d}_{max}}{\eta}}{2\frac{\|P\mathbf{b}\|}{b_{min}}(\|\mathbf{k}^*_x\| + b_q\psi)}\right)$,

for $q = q(0)$ if it is known (otherwise, for all $q \in \mathcal{Q}$), then

- *the adaptive system in* (1), (12), (15) *has uniformly ultimately bounded solutions and $|r(t)| \leq r^{\mathrm{I}}_{max,B}$ $\forall t > 0$, with $r^{\mathrm{I}}_{max,B}$ given in* (22),
- *the tracking error $\mathbf{e}(t)$ asymptotically decreases during each flow interval, i.e., when $\mathbf{x}_p(t) \in C_q$, and*
- *$|u_c(t)| \leq u_{max}$ and $|\dot{u}_{c,\rho}(t)| \leq \dot{u}_{max}$, i.e., control amplitude and rate limits are avoided $\forall t > 0$.*

Proof. For conciseness, only a sketch of the proof will be provided. The main idea for this proof is to ensure that the assumptions in [13, Theorem 5] for non-hybrid systems hold in spite of the jumps. The only assumption that may be violated by the introduction of jump dynamics is that of $r^{\mathrm{I}}_{max,B}$. This is avoided with the stricter bound on r^{I}_{max} in (22) and the dwell-time of τ^r_D in Lemma 5, which ensure that $r(t) < r^{\mathrm{I}}_{max,B}$ $\forall t > 0$. Thus, the results above follow immediately from [13, Theorem 5]. A minor straightforward modification of the proof in [13, Theorem 5] has also been carried out to make explicit the upper bound on Λ_r. □

Remark 2. *The dwell-time requirement (i.e., $\tau_D \geq \max\{\tau^V_D, \tau^r_D\}$) is a sufficient condition, rather than a prescriptive method to compute the dwell-time. This condition may be hard to guarantee* a priori*, especially since we assumed that mode transitions are autonomous. However, the dwell-time τ_D can be indirectly controlled by designing the open-loop reference model (ORM) to have a longer dwell-time. Since we have constructed Lyapunov functions that decrease monotonically with increasing dwell-time τ_D and with the assumption that the reset map $g_q(\mathbf{x}_p(t))$ is Lipschitz continuous for all $q \in \mathcal{Q}$, we can practically increase τ_D in simulation (indirectly via the ORM design) until satisfactory performance is attained.*

Remark 3. *As in [2], asymptotic tracking can be achieved, if Condition 3) in Proposition 1 is satisfied with strict inequality, by [20, Thm. 2.3]. This translates to the dwell-times in Theorems 1 and 2 being strictly greater than τ^V_D and $\max\{\tau^V_D, \tau^r_D\}$, respectively. In this paper, we opted for the weaker result of uniform ultimate boundedness so that the dwell-time requirement discussed above is more easily satisfied, and that our approach will be more generally applicable to a larger set of problems.*

IV. Illustrative Example

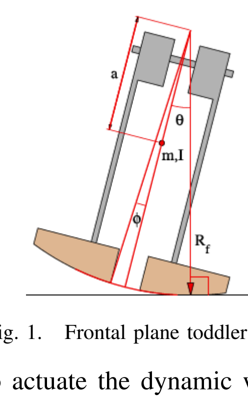


Fig. 1. Frontal plane toddler [22].

The goal is to actuate the dynamic walker, (cf. Figure 1 [22]), such that it toddles in a periodic fashion in the frontal plane, for which the continuous dynamics (assumed to be input-to-state stable) is given as:

$$H(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = \tau,$$

where τ is the control input/torque generated at the hip or from ankle actuation, and is amplitude and rate limited either by physical constraints or by choice, i.e., $|\tau| \leq \tau_{max}$ and $|\dot{\tau}| \leq \dot{\tau}_{max}$. When the curved portion of either foot is in contact with the ground, i.e., $|\theta| > \phi$, the dynamics are:

$$H(\theta) = I + ma^2 + mR_f^2 - 2mR_f a\cos\theta,$$
$$C(\theta,\dot{\theta}) = mR_f a\dot{\theta}\sin\theta,\ G(\theta) = mga\sin\theta,$$

whereas, when the ground contact is along the inside edge of the foot, i.e., $|\theta| \leq \phi$,

$$H(\theta) = I + ma^2 + mR_f^2 - 2mR_f a\cos\phi,$$
$$C(\theta,\dot{\theta}) = 0,\ G(\theta) = mg(a\sin\theta - R_f\sin\alpha),$$

where $\alpha = \theta - \phi$ if $\theta > 0$, otherwise $\alpha = \theta + \phi$. The mass is given by m, the moment of inertia by I and the lengths and angles are as depicted in Figure 1.

Furthermore, the swing leg collides with the ground when $\theta = 0$, and assuming an inelastic collision, the angular rate after collision, i.e., the jump dynamics, is given by

$$\dot{\theta}^+ = \dot{\theta}^- \cos\left[2\arctan\left(\frac{R_f\sin\phi}{R_f\cos\phi - a}\right)\right].$$

Thus, putting the continuous dynamics in the Brunovsky form given in (1), the hybrid adaptive system describing the frontal plane toddler model, as well as the chosen iCHARM dynamics can be written as follows:

Continuous dynamics (when $\theta_p \neq 0$):

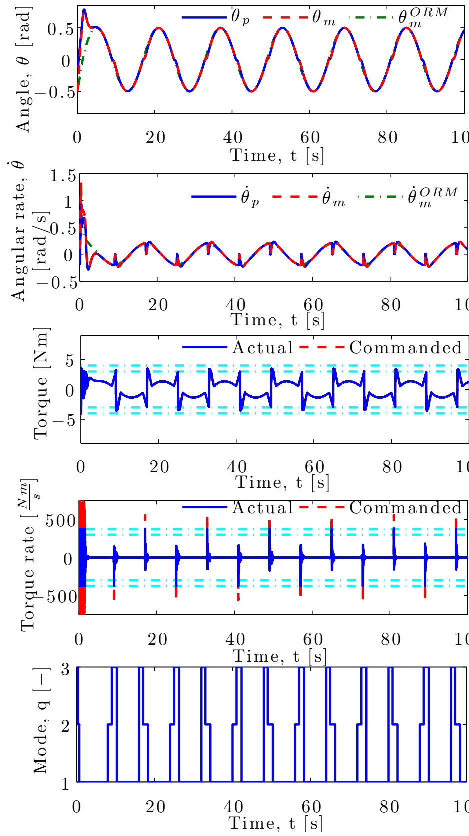
$$\ddot{\theta}_p = W_1 mR_f a\sin\theta_p\dot{\theta}_p^2 + W_2 mgR_f\sin\theta_p\cos\phi + W_3 mgR_f\cos\theta_p\sin\phi + b_q u + d,$$
$$\ddot{\theta}_m = -10\omega\dot{\theta}_m - 25\omega^2\theta_m + 26\omega^2 r + \hat{b}\Delta u_d + \mathrm{sgn}(e^T P\mathbf{b})d_{max},$$
$$\dot{r}_o = \dot{r}_d + \Lambda_r(r - r_d),$$
$$\dot{r} = \begin{cases} \dot{r}_o(t) + \frac{\hat{b}(t)}{26\omega^2}\Delta\dot{u}_d, & |u_c(t)| \leq u_{max}^{\delta_\mu}, \\ \dot{r}_o(t) + \frac{(1+\mu)\hat{b}(t)}{26\omega^2}\Delta\dot{u}_d, & u_{max}^{\delta_\mu} < |u_c(t)| \leq u_{max}, \\ \dot{r}_o(t), & \text{otherwise;} \end{cases}$$

Jump dynamics (when $\theta_p = 0$):

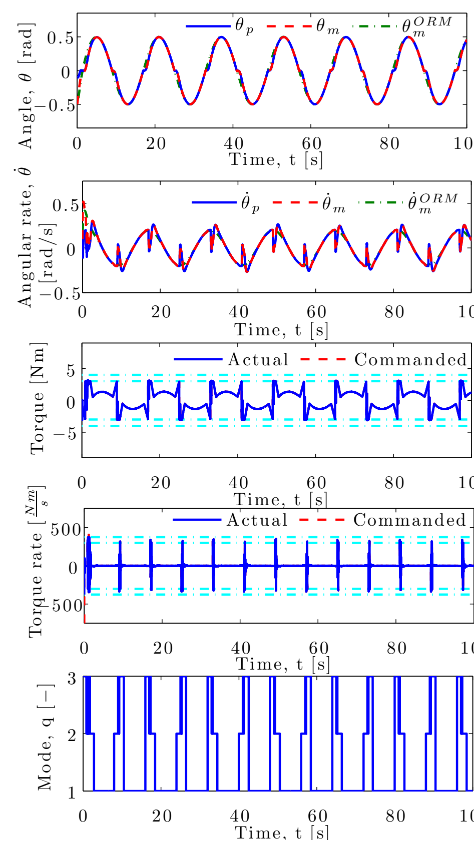
$$\dot{\theta}_p^+ = \dot{\theta}_p^- \cos\left[2\arctan\left(\frac{R_f\sin\phi}{R_f\cos\phi - a}\right)\right],$$
$$\dot{\theta}_m^+ = \dot{\theta}_m^- + \dot{\theta}_p^+ - \dot{\theta}_p^-,$$
$$r^+ = r(t) + \frac{1}{26\omega^2}(10\omega(\dot{\theta}_p^+ - \dot{\theta}_p) + \hat{W}_1(mR_f a\sin\theta_p(\dot{\theta}_p^+)^2 - mR_f a\sin\theta_p\dot{\theta}_p^2)),$$

where $u := \tau - mga\sin\theta_p$, and correspondingly, $u_{min} := \tau_{min} - mga\sin\theta_p$, $u_{max} := \tau_{max} - mga\sin\theta_p$, $\dot{u}_{min} := \dot{\tau}_{min} - mga\dot{\theta}_p\cos\theta_p$ and $\dot{u}_{max} := \dot{\tau}_{max} - mga\dot{\theta}_p\cos\theta_p$. When $q = 1$ ($|\theta_p| > \phi$), $W_1 = -\frac{1}{a_1}$, $W_2 = 0$, $W_3 = 0$ and $b_q = \frac{1}{a_1}$; when $q = 2$ ($0 < \theta_p \leq \phi$), $W_1 = 0$, $W_2 = \frac{1}{a_2}$, $W_3 = -\frac{1}{a_2}$ and $b_q = \frac{1}{a_2}$; and when $q = 3$ ($-\phi \leq \theta_p \leq 0$), $W_1 = 0$, $W_2 = \frac{1}{a_2}$, $W_3 = \frac{1}{a_2}$ and $b_q = \frac{1}{a_2}$, with $a_1 = I + ma^2 + mR_f^2$, and $a_2 = I + ma^2 + mR_f^2 - 2mR_f a\cos\phi$.

Note that the $\frac{2mR_f a\cos\theta_p\ddot{\theta}_p}{a_1}$ term in $q = 1$ is treated as a disturbance, $|d| \leq d_{max} := \frac{2mR_f a\ddot{\theta}_{max}}{a_1}$, where $\ddot{\theta}_{max}$ is the maximum expected $|\ddot{\theta}_p|$ for the entire trajectory. The desired

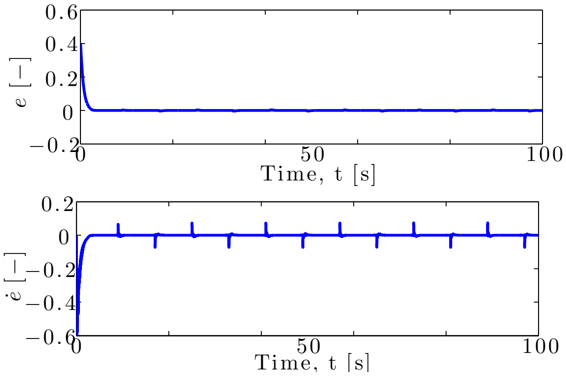


(a) $\mu = 0.1$, $\rho = 1$

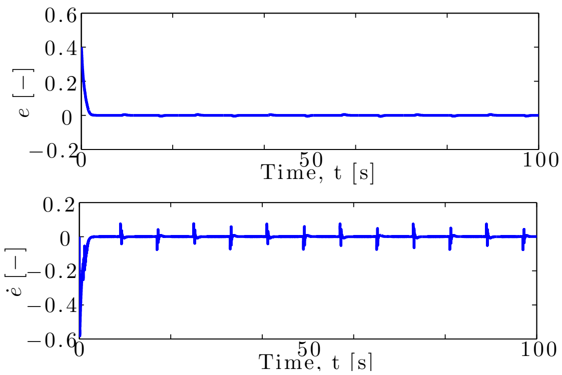


(b) $\mu = 25$, $\rho = 10$

Fig. 2. θ, $\dot{\theta}$, inputs, input rates and modes.



(a) $\mu = 0.1$, $\rho = 1$



(b) $\mu = 25$, $\rho = 10$

Fig. 3. Time history of errors in θ and $\dot{\theta}$.

reference signal is given by $r_d = r_d^{max}\sin(\omega t)$ and $\omega = \frac{2\pi}{T}$, where T is the desired period. Figure 2 shows the simulation of this adaptive system with: $a = 0.1\ m$, $R_f = 0.5\ m$, $m = 3\ kg$, $I = 0.1\ kgm^2$, $\phi = 0.2\ rad$, $g = 9.81\ ms^{-2}$, $|\tau| \leq 4\ Nm$, $|\dot{\tau}| \leq 375 Nms^{-1}$, $\delta_\mu = 0.25\tau_{max}$, $\delta_\rho = 0.2\dot{\tau}_{max}$, $T = 16\ s$, $r_d^{max} = 0.5$, $\frac{1}{a_2} \leq \hat{b}(t) \leq \frac{1}{a_2}$, $\Lambda_r = -10$, $Q = diag(240, 20)$, $\gamma_b = 50$, $\Gamma_W = diag(100, 100, 100)$, $\ddot{\theta}_{max} = \omega^2 r_d^{max} = 0.0771\ rads^{-2}$, $\theta_p(0) = -0.1$, $\dot{\theta}_p(0) = 0.1$, $\theta_m(0) = -0.5$, $\dot{\theta}_m(0) = 0.1$, $r(0) = r_d(0) = 0$, $\hat{b}(0) = \frac{1}{a_2}$, $\hat{W}_1(0) = 0$, $\hat{W}_2(0) = \frac{1}{a_2}$, $\hat{W}_3(0) = \frac{1}{a_2}$ and $u_d(0) = u_d^\star(0)$.

We see from Figure 2 that the system states track the desired trajectories satisfactorily even when subjected to the input amplitude and rate constraints. Besides, when μ and ρ are chosen to be sufficiently large, the commanded input amplitude and rate are observed to be within their bounds. This larger choice of parameters only results in a marginally different state trajectory and also slightly higher tracking errors (differences are almost imperceptible in Figure 3). In both cases, the tracking errors remain bounded, as desired.

V. Conclusion

This paper proposed an adaptive control approach to track a modified reference model (iCHARM) in an uniformly ultimately bounded manner when an uncertain hidden mode hybrid system in Brunovsky form is subject to input amplitude and rate constraints as well as bounded disturbances. For input-to-state stable systems with constrained input amplitude and rate, we showed that tracking can be achieved in a stable manner globally even in the presence of bounded disturbances, whereas for input-to-state unstable systems, we characterized their local regions of attraction. Moreover, we provided an approach to prevent input amplitude and rate saturation via (ρ, μ)-modification. By means of a numerical example, we illustrated the effectiveness of our approach for input-constrained hidden mode tracking.

References

[1] R. Verma and D. Del Vecchio. Safety control of hidden mode hybrid systems. *IEEE Transactions on Automatic Control*, 57(1):62–77, 2012.

[2] S.Z. Yong and E. Frazzoli. Hidden mode tracking control for a class of hybrid systems. In *Proceedings of the American Control Conference*, pages 5735–5741, Jun 2013.

[3] D.S. Bernstein and A.N. Michel. A chronological bibliography on saturating actuators. *International Journal of Robust and Nonlinear Control*, 5(5):375–380, 1995.

[4] T. Hu and Z. Lin. *Control systems with actuator saturation: analysis and design*. Springer Science & Business Media, 2001.

[5] S. Tarbouriech and G. Garcia. *Control of uncertain systems with bounded inputs*. Springer-Verlag New York, Inc., 1997.

[6] R.V. Monopoli. Adaptive control for systems with hard saturation. In *IEEE Conference on Decision and Control*, volume 14, pages 841–843, 1975.

[7] S.P. Karason and A.M. Annaswamy. Adaptive control in the presence of input constraints. *IEEE Transactions on Automatic Control*, 39(11):2325–2330, 1994.

[8] E. Lavretsky and N. Hovakimyan. Positive μ-modification for stable adaptation in the presence of input constraints. In *Proceedings of the American Control Conference*, volume 3, pages 2545–2550, 2004.

[9] E. Lavretsky and N. Hovakimyan. Positive μ-modification for stable adaptation in dynamic inversion based adaptive control with input saturation. In *Proceedings of the American Control Conference*, volume 5, pages 3373–3378, 2005.

[10] A. Leonessa, M.M. Haddad, and T. Hayakawa. Adaptive tracking for nonlinear systems with control constraints. In *Proceedings of the American Control Conference*, volume 2, pages 1292–1297, 2001.

[11] J. Farrell, M. Polycarpou, and M. Sharma. On-line approximation based control of uncertain nonlinear systems with magnitude, rate and bandwidth constraints on the states and actuators. In *Proceedings of the American Control Conference*, volume 3, pages 2557–2562, 2004.

[12] A. Leonessa, W.M. Haddad, T. Hayakawa, and Y. Morel. Adaptive control for nonlinear uncertain systems with actuator amplitude and rate saturation constraints. *International Journal of Adaptive Control and Signal Processing*, 23(1):73–96, 2009.

[13] S.Z. Yong and E. Frazzoli. Asymptotic adaptive tracking with input amplitude and rate constraints and bounded disturbances. In *IEEE Conference on Decision and Control*, pages 1256–1263, Dec 2014.

[14] L. Menini and A. Tornambe. Asymptotic tracking of periodic trajectories for a simple mechanical system subject to nonsmooth impacts. *IEEE Transactions on Automatic Control*, 46(7):1122–1126, Jul 2001.

[15] R.G. Sanfelice, J.J.B. Biemond, N. van de Wouw, and W. P.M.H. Heemels. Tracking control for hybrid systems via embedding of known reference trajectories. In *Proceedings of the American Control Conference*, pages 869–874, Jun 2011.

[16] M. Robles and R.G. Sanfelice. Hybrid controllers for tracking of impulsive reference state trajectories: a hybrid exosystem approach. In *Proceedings of the 14th International Conference on Hybrid Systems: Control and Computation*, pages 231–240, 2011.

[17] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, Apr 2009.

[18] D. Liberzon. *Switching in systems and control*. Springer, 2003.

[19] H.K. Khalil. *Nonlinear systems*. Prentice Hall New Jersey, 3rd edition, 1996.

[20] M.S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, 1998.

[21] J.J.E. Slotine and W. Li. *Applied Nonlinear Control*. Prentice-Hall, 1991.

[22] R. Tedrake, T.W. Zhang, M. Fong, and H.S. Seung. Actuating a simple 3D passive dynamic walker. In *Proceedings of the International Conference on Robotics and Automation*, pages 4656–4661, 2004.